Boundary controllability for the quasilinear wave equation

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Abstract We study the boundary exact controllability for the quasilinear wave equation in the higher-dimensional case. Our main tool is the geometric analysis. We derive the existence of long time solutions near an equilibrium, prove the locally exact controllability around the equilibrium under some checkable geometrical conditions. We then establish the globally exact controllability in such a way that the state of the quasilinear wave equation moves from an equilibrium in one location to an equilibrium in another location under some geometrical condition. The Dirichlet action and the Neumann action are studied, respectively. Our results show that exact controllability is geometrical characters of a Riemannian metric, given by the coefficients and equilibria of the quasilinear wave equation. A criterion of exact controllability is given, which based on the sectional curvature of the Riemann metric. Some examples are presented to verify the global exact controllability.

Keywords quasi-linear wave equation, exact controllability, sectional curvature

AMS(MOS) subject classifications 49B, 49E, 35B35, 35L65, 35L70, 38J45

1 Introduction and the main results

Let $\Omega \subset \mathcal{R}^n$ be an open, bounded set with the smooth boundary Γ. Suppose that Γ consists of two disjoint parts, Γ_0 and Γ_1. Let $T > 0$ be given. We consider a controllability

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problem

$$\begin{cases} \ddot{u} = \sum_{ij=1}^n a_{ij}(x, \nabla u) u_{x_i x_j} + b(x, \nabla u) \quad \text{on} \quad (0,T) \times \Omega, \\ u = 0 \quad \text{on} \quad (0,T) \times \Gamma_1, \\ u = \varphi \quad \text{on} \quad (0,T) \times \Gamma_0, \\ u(0) = u_0, \quad \dot{u}(0) = u_1, \end{cases} \tag{1.1}$$

where $a_{ij}(x,y)$, $b(x,y)$ are smooth functions on $\overline{\Omega} \times \mathcal{R}^n$ such that

$$A(x,y) = (a_{ij}(x,y)) > 0 \quad \forall\ (x,y) \in \overline{\Omega} \times \mathcal{R}^n, \tag{1.2}$$

$$b(x,0) = 0 \quad \forall\ x \in \overline{\Omega}. \tag{1.3}$$

Let u_0, u_1, $\hat{u}_0$, and $\hat{u}_1$ be given functions on $\overline{\Omega}$ and $T > 0$ be given. If there is a boundary function φ on $(0,T) \times \Gamma_0$ such that the solution of the problem (1.1) satisfies

$$u(T) = \hat{u}_0, \quad \dot{u}(T) = \hat{u}_1 \quad \text{on} \quad \Omega,$$

we say the system (1.1) is exactly controllable from (u_0, u_1) to $(\hat{u}_0, \hat{u}_1)$ at time T by boundary with the Dirichlet action.

In the case of one dimension, these problems have been studied by Cirina [4], Li and Rao [16], Schmidt [19], and so on. In the case of multi-dimension, $n \geq 2$, very little is known in the content of control. The work here represents a substantial advance on this topic. The key issue is to establish the geometrical structure of the problem: The locally exact controllability is equivalent to the smooth control problem of a linear, variable coefficient wave equation which is related to the geometric theory. The detail study of the smooth control of the linear problem provides a smooth control to the quasilinear problem. Then a compactness principle gives the globally exact controllability. This idea is also used to study the existence of global solutions of the quasilinear wave equation with boundary dissipation by Yao [26].

Let us choose some Sobolev spaces to formulate our problems. Let

$$m \geq [n/2] + 3$$

be a given positive integer. Inspired by Dafermos and Hrusa [7], we assume initial data $(u_0, u_1) \in H^m(\Omega) \times H^{m-1}(\Omega)$ to study the possibility of moving it to another state in $H^m(\Omega) \times H^{m-1}(\Omega)$ at time T via a boundary control $\varphi \in \cap_{k=0}^{m-2} C^k \left([0,T], H^{m-k-1/2}(\Gamma_0)\right)$.

In general, solutions of the system (1.1) may below up in a finite time even if the initial data and the boundary control are smooth. On the other hand, in order to move one state to another, the control time must be larger than the wave length of the system. To cope with those situations, we shall study the locally exact controllability of the system around an equilibrium and the globally exact controllability form one equilibrium to another.

We say $w \in H^m(\Omega)$ is an equilibrium of the system (1.1) if

$$\sum_{ij=1}^{n} a_{ij}(x, \nabla w) w_{x_i x_j} + b(x, \nabla w) = 0 \quad \text{on} \quad \Omega. \tag{1.4}$$

We say that $(u_0, u_1) \in H^m(\Omega) \times H^{m-1}(\Omega)$ and $\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-1/2}(\Gamma_0)\right)$ satisfy the compatibility conditions of m order if

$$u_k \in H^{m-k}(\Omega), \quad u_k|_{\Gamma_1} = 0, \tag{1.5}$$

$$\varphi^{(k)}(0) = u_k|_{\Gamma_0}, \quad k = 0,\, 1,\, \cdots,\, m-1,$$

where for $k \geq 2$,

$$u_k = u^{(k)}(0), \tag{1.6}$$

as computed formally (and recursively) in terms of u_0 and u_1, using the equation in (1.1).

Let

$$H^1_{\Gamma_1}(\Omega) = \{\, v \,|\, v \in H^1(\Omega),\ v|_{\Gamma_1} = 0 \,\}. \tag{1.7}$$

Near one equilibrium, the system has solutions of long time. This is the following

Theorem 1.1 *Let $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be an equilibrium of the problem* (1.1)*. Let $T > 0$ be arbitrary given. Then there is $\varepsilon_T > 0$, which depends on the time T, such that, if $(u_0, u_1) \in \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^{m-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right)$ satisfy*

$$\|u_0 - w\|_m < \varepsilon_T, \quad \|u_1\|_{m-1} < \varepsilon_T,$$

where $\|\cdot\|_m$ denotes the norm of $H^m(\Omega)$, and $\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-1/2}(\Gamma_0)\right)$ with $\varphi^{(k)} \in H^1\left((0,T) \times \Gamma_0\right)$ for $0 \leq k \leq m-1$ satisfies the compatibility conditions with (u_0, u_1) of m order and

$$\sum_{k=0}^{m-2} \|\hat{\varphi}^{(k)}\|^2_{C\left([0,T], H^{m-1/2-k}(\Gamma_0)\right)} + \sum_{k=0}^{m-1} \|\hat{\varphi}^{(k)}\|^2_{H^1((0,T)\times\Gamma_0)} < \varepsilon_T,$$

where

$$\hat{\varphi}(t,x) = \varphi - w|_{\Gamma_0},$$

then the system (1.1) *has a solution*

$$u \in \cap_{k=0}^{m} C^k\left([0,T], H^{m-k}(\Omega)\right). \tag{1.8}$$

Let $w \in H^m_{\Gamma_1}(\Omega)$ be an equilibrium of the system (1.1). We define

$$g = A^{-1}(x, \nabla w) \tag{1.9}$$

as a Riemannian metric on $\overline{\Omega}$ and consider the couple $(\overline{\Omega}, g)$ as a Riemannian manifold with a boundary Γ. Here the metric g depends on the functions $a_{ij}(\cdot,\cdot)$ and also on the

equilibrium w. We denote by $\langle\cdot,\cdot\rangle_g$ the inner product induced by g. Let $x^0 \in \overline{\Omega}$ be given. We denote by $\rho(x) = \rho(x, x^0)$ the distance function from $x \in \overline{\Omega}$ to x^0 under the Riemannian metric g.

Definition An equilibrium $w \in H^m(\Omega)$ is called exactly controllable if there are $x^0 \in \overline{\Omega}$ and $\rho_0 > 0$ such that

$$D_g^2 \rho^2(X, X) \geq \rho_0 |X|_g^2 \quad \forall\, X \in \Omega_x,\ x \in \overline{\Omega}, \tag{1.10}$$

where $D_g^2\rho^2$ denotes the Hessian of the function ρ^2 under the metric g which is a bilinear form on $\overline{\Omega}$.

The condition (1.10) means that the function $\rho^2(x)$ is strictly convex on $\overline{\Omega}$ under the metric g. This is true if x is in a neighbourhood of x^0. Whether it holds on the whole domina $\overline{\Omega}$ is closely related to the sectional curvature of the Riemannian metric g, see some examples later. Yao [24] presents a counterexample where the condition (1.10) is not always true for all $x \in \overline{\Omega}$ even when $A(x, y) = A(x)$ (the linear problem). If the matrices $A(x, y) = A(y)$ and the equilibrium is zero, the condition (1.10) holds for any $\Omega \subset \mathcal{R}^n$ with $\rho_0 = 2$. A proposition below is useful to verify the condition (1.10).

For $x \in \overline{\Omega}$, let $\Pi \subset R_x^n$ be a two-dimensional subspace. Denote by $k_x(\Pi)$ the sectional curvature of the subspace Π at x under the Riemannian metric g. Let

$$\kappa = \sup_{x \in \overline{\Omega},\, \Pi \subset \mathcal{R}_x^n} k_x(\Pi). \tag{1.11}$$

Then

Proposition 1.1 *If an equilibrium w is such that $\kappa \leq 0$, then w is exactly controllable. Suppose $\kappa > 0$. Set*

$$\lambda = \inf_{x \in \overline{\Omega},\, y \in \mathcal{R}^n,\, |y|=1} \sqrt{\langle A(x, \nabla w) y,\ y\rangle}.$$

If there is a point $x_0 \in \overline{\Omega}$ such that

$$\overline{\Omega} \subset B\left(x_0, \frac{\lambda\pi}{2\sqrt{\kappa}}\right), \tag{1.12}$$

where $B\left(x_0, \frac{\lambda\pi}{2\sqrt{\kappa}}\right) = \{\, x \,|\, x \in \mathcal{R}^n,\ |x - x_0| < \frac{\lambda\pi}{2\sqrt{\kappa}}\,\}$, then w is exactly controllable.

Near one equilibrium being exactly controllable, we have the following exact controllability results:

Theorem 1.2 *Let an equilibrium* $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ *be exactly controllable. Let*

$$T_0 = \frac{4}{\rho_0} \sup_{x \in \overline{\Omega}} \rho. \tag{1.13}$$

Furthermore, if $\Gamma_1 \neq \emptyset$, *we assume that*

$$\rho_\nu \leq 0 \quad \forall\, x \in \Gamma_1, \tag{1.14}$$

where ρ_ν *is the normal derivative of the distance function* ρ *of the metric* g *with respect to the normal* ν *of the dot metric of* $\mathcal{R}^n$. *Then, for* $T > T_0$ *given, there is* $\varepsilon_T > 0$ *such that, for any* $(u_0^i, u_1^i) \in H^m(\Omega) \times H^{m-1}(\Omega)$ *with*

$$\|u_0^i - w\|_m < \varepsilon_T, \quad \|u_1^i\|_{m-1} < \varepsilon_T, \quad u_0^i|_{\Gamma_1} = u_1^i|_{\Gamma_1} = 0, \quad i = 1,\, 2,$$

we can find $\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-1/2}(\Gamma_0)\right)$ *with* $\varphi^{(k)} \in H^1\left((0,T) \times \Gamma_0\right)$ *for* $0 \leq k \leq m-1$ *which is compatible with* (u_0^1, u_1^1) *of* m *order such that the solution of the system* (1.1) *with the initial data* (u_0^1, u_1^1) *satisfies*

$$u(T) = u_0^2, \quad \dot{u}(T) = u_1^2. \tag{1.15}$$

The above is a local result. However, if we have enough equilibria exactly controllable, we can move the quasilinear wave state along a curve of equilibria, moving in successive small steps from one equilibrium to another nearby equilibrium until the target equilibrium is reached. This uses the open mapping theorem, locally exact controllability, and a compactness argument. This approach was used by Schmidt [19] for the quasilinear string.

Let $w \in H^m_{\Gamma_1}(\Omega)$ be a given equilibrium. For $\alpha \in [0,1]$, we assume that $w_\alpha \in H^m(\Omega)$ are the solutions of the Dirichlet problem

$$\begin{cases} \sum_{ij=1}^n a_{ij}(x, \nabla w_\alpha) w_{\alpha x_i x_j} + b(x, \nabla w_\alpha) = 0 \quad x \in \Omega, \\ w_\alpha|_\Gamma = \alpha w|_\Gamma, \end{cases} \tag{1.16}$$

such that

$$\sup_{\alpha \in [0,1]} \|w_\alpha\|_m < \infty. \tag{1.17}$$

For the existence of the classical solution to the Dirchlet problem (1.16), for example, see Gilbarg and Trudinger [9].

Theorem 1.3 *Let an equilibrium* $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ *be exactly controllable. Let* $w_\alpha \in H^m(\Omega)$, *given by* (1.16), *be also exactly controllable for all* $\alpha \in [0,1]$ *such that* (1.17) *hold. Then, there are* $T > 0$ *and*

$$\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-1/2}(\Gamma_0)\right)$$

with $\varphi^{(k)} \in H^1\left((0,T) \times \Gamma_0\right)$ *for* $0 \leq k \leq m-1$ *which is compatible with the initial data* $(w, 0)$ *such that the solution of the system* (1.1) *with* $(u_0, u_1) = (w, 0)$ *satisfies*

$$u(T) = \dot{u}(T) = 0.$$

Since the quasilinear wave equation is time-reversible, an equilibrium can be moved to another if they can both be moved to zero. However, this result only gives the existence of the control time T. We do not know how large the T is because it is given by the compactness principle.

Next, we turn to the boundary control with the Neumann action. Let $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ with Γ_1 nonempty. This time we assume that the quasilinear part of the system is in the divergence form. Let $T > 0$ be given. We consider a controllability problem

$$\begin{cases} \ddot{u} = \operatorname{div} \mathbf{a}(x, \nabla u) \quad \text{on} \quad (0,T) \times \Omega, \\ u = 0 \quad \text{on} \quad (0,T) \times \Gamma_1, \\ \langle \mathbf{a}(x, \nabla u),\, \nu \rangle = \varphi \quad \text{on} \quad (0,T) \times \Gamma_0, \\ u(0) = u_0, \quad \dot{u}(0) = u_1, \end{cases} \tag{1.18}$$

where $\mathbf{a}(\cdot,\cdot) = (a_1(\cdot,\cdot), \cdots, a_n(\cdot,\cdot))$ and $a_i(\cdot,\cdot)$ are smooth functions on $\overline{\Omega} \times \mathcal{R}^n$ such that

$$\mathbf{a}(x, 0) = 0 \quad \forall\, x \in \overline{\Omega}; \quad A(x,y) = \Big(a_{iy_j}(x,y) \Big) > 0 \quad \forall\ (x,y) \in \overline{\Omega} \times \mathcal{R}^n. \tag{1.19}$$

In the problem (1.18), ν is the normal of the boundary Γ in the dot metric of $\mathcal{R}^n$.

We say $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ is an equilibrium of the system (1.18) if

$$\operatorname{div} \mathbf{a}(x, \nabla w) = 0 \quad \text{on} \quad \Omega. \tag{1.20}$$

We say that $(u_0, u_1) \in H^m(\Omega) \times H^{m-1}(\Omega)$ and $\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-3/2}(\Gamma_0)\right)$ satisfy the compatibility conditions of m order with the Neumann boundary data on Γ_0 and the Dirichlet data on Γ_1 if (1.5) hold and

$$\varphi^{(k)}(0) = \begin{cases} \langle \mathbf{a}(x, \nabla u_0),\, \nu \rangle \quad x \in \Gamma_0, \quad k = 0, \\ \langle A(x, \nabla u_0) \nabla u_k,\, \nu \rangle \quad x \in \Gamma_0, \quad 1 \le k \le m-1, \end{cases} \tag{1.21}$$

where for $k \ge 2$, u_k are given by (1.6).

Theorem 1.4 *Let $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be an equilibrium of the problem* (1.18)*. Let $T > 0$ be arbitrary given. Then there is $\varepsilon_T > 0$, which depends on the time T, such that, if $(u_0, u_1) \in \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^{m-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right)$ satisfy*

$$\|u_0 - w\|_m < \varepsilon_T, \quad \|u_1\|_{m-1} < \varepsilon_T,$$

and $\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-3/2}(\Gamma_0)\right)$ is such that $\varphi^{(k)} \in L^2\left((0,T), H^{1/2}(\Gamma_0)\right)$ for $0 \le k \le m-1$, which satisfies the compatibility conditions (1.21) *with (u_0, u_1) of m order and*

$$\sum_{k=0}^{m-2} \|\hat{\varphi}^{(k)}\|^2_{C\left([0,T], H^{m-k-3/2}(\Gamma_0)\right)} + \sum_{k=0}^{m-1} \|\hat{\varphi}^{(k)}\|^2_{L^2\left((0,T), H^{1/2}(\Gamma_0)\right)} < \varepsilon_T,$$

where

$$\hat{\varphi}(t,x) = \varphi - \langle \boldsymbol{a}(x, \nabla w),\, \nu\rangle \quad x \in \Gamma_0,$$

then the system (1.18) *has a solution*

$$u \in \cap_{k=0}^{m} C^k\left([0,T], H^{m-k}(\Omega)\right). \tag{1.22}$$

Theorem 1.5 *Let an equilibrium $w \in H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable. Let*

$$\rho_\nu \leq 0 \quad \forall\, x \in \Gamma_1. \tag{1.23}$$

Then there exists a $T_0 > 0$ such that the following things are true. For any $T > T_0$ given, there is $\varepsilon_T > 0$ such that, for any $(u_0^i, u_1^i) \in H^{m+1}(\Omega) \times H^m(\Omega)$ with

$$\|u_0^i - w\|_{m+1} < \varepsilon_T, \quad \|u_1^i\|_m < \varepsilon_T, \quad u_0^i|_{\Gamma_1} = u_1^i|_{\Gamma_1} = 0, \quad i = 1,\, 2,$$

we can find $\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-3/2}(\Gamma_0)\right)$ with $\varphi^{(k)} \in L^2\left((0,T), H^{1/2}(\Gamma_0)\right)$ for $0 \leq k \leq m-1$ which is compatible with (u_0^1, u_1^1) of m order such that the solution of the system (1.18) *with the initial data (u_0^1, u_1^1) satisfies*

$$u(T) = u_0^2, \quad \dot{u}(T) = u_1^2. \tag{1.24}$$

Here we lose an explicit formula of T_0.

Unlike the control with the Dirichlet action, we only have the exact controllability results in the space $H^{m+1}(\Omega) \times H^m(\Omega)$ by a control $\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-3/2}(\Gamma_0)\right)$ with $\varphi^{(k)} \in L^2\left((0,T), H^{1/2}(\Gamma_0)\right)$ for $0 \leq k \leq m-1$. This is because the Neumann action loses a regularity of 1 order (actually, 1/2 order), see Theorem 2.2 in the end of Section 2. In addition, although we can move one state to another in the space $H^{m+1}(\Omega) \times H^m(\Omega)$, we can not guarantee the solution $(u(t), \dot{u}(t))$ of the problem (1.18) always stays in $H^{m+1}(\Omega) \times H^m(\Omega)$ in the process of the control for $0 \leq t \leq T$ where they are actually in the space $H^m(\Omega) \times H^{m-1}(\Omega)$ for all $t \in [0,T]$ by Theorem 1.4. The same things happen to the globally exact controllability results in Theorem 1.6 below.

Let an equilibrium $w \in H^{m+1}_{\Gamma_1}(\Omega)$ be given. For $\alpha \in [0,1]$, we assume that $w_\alpha \in H^{m+1}(\Omega)$ are the solutions of the Dirichlet problem (1.16) with, this time, an uniform bound

$$\sup_{\alpha \in [0,1]} \|w_\alpha\|_{m+1} < \infty. \tag{1.25}$$

Theorem 1.6 *Let an equilibrium $w \in H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable. Let $w_\alpha \in H^{m+1}(\Omega)$ be also exactly controllable for all $\alpha \in [0,1]$ such that* (1.25) *hold. Then, there are $T > 0$ and*

$$\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-3/2}(\Gamma_0)\right)$$

with $\varphi^{(k)} \in L^2\left((0,T), H^{1/2}(\Gamma_0)\right)$ *for* $0 \leq k \leq m-1$, *which is compatible with the initial data* $(w, 0)$ *of* m *order such that the solution of the system* (1.18) *with* $(u_0, u_1) = (w, 0)$ *satisfies*

$$u(T) = \dot{u}(T) = 0.$$

Boundary exact controllability on linear problems has been developing since 70's and very active in recent years. We mention Bardos, Lebeau, Rauch [2], Castro, Zuazua [3], Egorov [8], Fattorini [11], Ho [12], Lasiecka, Triggiani [13], Lions [17], Russel [18], Seidman [20], Tataru [21], Yao [24], [25], Yong, Zhang [27], just a few.

Finally, let us see some examples to verify Theorem 1.3.

Example 1.1 *Let* $n = 2$ *and* $m = 4$. *Consider the control problem*

$$\begin{cases} \ddot{u} = (|\nabla u|^2 + 1)\Delta u \quad (t,x) \in (0,T) \times \Gamma, \\ u|_\Gamma = \varphi \quad 0 \leq t \leq T, \\ u(0) = w \quad \dot{u}(0) = 0 \quad x \in \Omega, \end{cases} \tag{1.26}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

Then $w \in H^4(\Omega)$ *is an equilibrium if and only if*

$$\Delta w = 0 \quad x \in \Omega.$$

Let $w \in H^4(\Omega)$ *be an equilibrium. Then metric* (1.9) *is given by*

$$g = A^{-1}(x, \nabla w), \quad A(x, \nabla w) = \begin{pmatrix} |\nabla w|^2 + 1 & 0 \\ 0 & |\nabla w|^2 + 1 \end{pmatrix}.$$

By Lemma 3.2, Yao [24], the Gauss curvature of the Riemmannin manifold $(\overline{\Omega}, g)$ *is*

$$k(x) = |D^2 w|^2, \quad x \in \overline{\Omega},$$

and

$$\kappa = \sup_{x \in \overline{\Omega}} |D^2 w|^2,$$

where $D^2 w$ *is the Hessian of* w *in the dot metric of* $\mathcal{R}^2$. **Then the zero equilibrium,** $w = 0$**, is exactly controllable for any** $\Omega \subset \mathcal{R}^n$. *In addition, we have the conclusion:* **If two equilibria** $w_i \neq 0$ **in** $H^4(\Omega)$ **are such that there are** $x_i \in \overline{\Omega}$ **satisfying**

$$\overline{\Omega} \subset B(x_i, \gamma_i), \tag{1.27}$$

where

$$\gamma_i = \frac{\pi\sqrt{1 + \inf_{x \in \overline{\Omega}} |\nabla w_i|^2}}{2 \sup_{x \in \overline{\Omega}} |D^2 w_i|}, \tag{1.28}$$

for $i = 1$**,** 2**, then there are a control time** $T > 0$ **and a control function**

$$\varphi \in \cap_{k=0}^{2} C^k \left([0, T], H^{7/2-k}(\Gamma) \right)$$

with $\varphi^{(k)} \in H^1((0,T) \times \Gamma)$ **for** $0 \le k \le 3$ **such that the solution of the problem** (1.26) **with the initial** $(w_1, 0)$ **satisfies**

$$u(T) = w_2, \quad \dot{u}(T) = 0.$$

Suppose that the two equilibria w_i *in* $H^4(\Omega)$ *are such that the conditions* (1.27) *are true. Then, for* $\alpha \in [0,1]$*,* $w_{i\alpha} = \alpha w_i$ *are equilibria with* $w_{i\alpha}|_\Gamma = \alpha w_i|_\Gamma$*. Since*

$$\gamma_i \le \frac{\pi \sqrt{1 + \alpha^2 \inf_{x \in \overline{\Omega}} |\nabla w_i|^2}}{2\alpha \sup_{x \in \overline{\Omega}} |D^2 w_i|},$$

for all $0 < \alpha \le 1$*, the conditions* (1.27) *are true for all* $w_{i\alpha}$ *with* $a \in [0,1]$*. By Theorem 1.3 and Proposition 1.1, the initial data* $(w_i, 0)$ *can be moved to* $(0,0)$*, respectively.*

Let

$$w_1 = a(x^2 - y^2), \quad w_2 = axy, \quad \Omega = \textit{the unit disc},$$

where $0 < a < 1/(2\sqrt{2})$*. It is easy to check that* w_i *meet the conditions* (1.27) *for* $i = 1$*,* 2*. Then the state of the system* (1.26) *can be moved from* $(w_1, 0)$ *to* $(w_2, 0)$ *at some time* $T > 0$*.*

Example 1.2 *Consider the control problem*

$$\begin{cases} \ddot{u} = (|\nabla u|^2 + 1)^{-1} \Delta u & (t,x) \in (0,T) \times \Gamma, \\ u|_\Gamma = \varphi & 0 \le t \le T, \\ u(0) = w \quad \dot{u}(0) = 0 & x \in \Omega. \end{cases} \tag{1.29}$$

Let $w \in H^4(\Omega)$ *be an equilibrium. The metric is*

$$g = \begin{pmatrix} |\nabla w|^2 + 1 & 0 \\ 0 & |\nabla w|^2 + 1 \end{pmatrix}.$$

The Gauss curvature of $(\overline{\Omega}, g)$ *is*

$$k(x) = -\frac{|D^2 w|^2}{(|\nabla w|^2 + 1)^3} \le 0, \quad \forall\, x \in \overline{\Omega}.$$

We have the conclusion: **For any two equilibria** w_1**,** $w_2 \in H^4(\Omega)$ **and any** $\Omega \subset \mathcal{R}^n$**, there are a control time** $T > 0$ **and a control function** $\varphi \in \cap_{k=0}^{2} C^k \left([0,T], H^{7/2-k}(\Gamma) \right)$ **with** $\varphi^{(k)} \in H^1((0,T) \times \Gamma)$ **for** $0 \le k \le 3$ **such that the state of the system** (1.29) **is moved from** $(w_1, 0)$ **to** $(w_2, 0)$.

2 Solutions of long time

The basic results of the existence of short time solutions to the quasilinear wave equation has been established by Dafermos and Hrusa [7]. We here only study some energy estimates of the short time solutions to have long time solutions when initial data are close to an equilibrium.

Let $(u_0, u_1) \in H^m(\Omega) \times H^{m-1}(\Omega)$ and $\varphi \in C^\infty((0,T) \times \Gamma_0)$ satisfy the compatibility conditions of m order. If we extend φ from $(0,T) \times \Gamma_0$ to $(0,T) \times \Omega$, still denoted by φ and let

$$v = u - \varphi \tag{2.1}$$

as a new unknown, then the problem will have solutions $v \in \cap_{k=0}^m C^k\left([0,T], H^{m-k}(\Omega)\right)$ of short time by Dafermos and Hrusa [7], Theorem 5.1.

To obtain solutions of long time near an equilibrium location, we need to estimate the energy of solutions to the problem (1.1). We observe that, if we apply Dafermos and Hrusa [7], Theorem 3.1 to our problem after the transform (2.1), we shall see that the regularity of $\varphi \in \cap_{k=0}^{m-1} C^k\left([0,T], H^{m-k-1/2}(\Gamma_0)\right)$ is insufficient to guarantee $u \in \cap_{k=0}^m C^k\left([0,T], H^{m-k}(\Omega)\right)$ because we have lost a regularity of $1/2$ order by the transform (2.1). For this reason, we shall here work out our energy estimates starting from the problem (1.1) directly.

We suppose that the equilibrium is the zero, $w = 0$, in this section. If an equilibrium $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ is not zero, we can make an transform by

$$u = w + v,$$

and consider the v-problem

$$\begin{cases} \ddot{v} = \sum_{ij} \hat{a}_{ij}(x, \nabla v) v_{x_i x_j} + \hat{b}(x, \nabla v) \quad (t,x) \in (0,T) \times \Omega, \\ v|_{\Gamma_1} = 0, \quad v|_{\Gamma_0} = \varphi - w|_{\Gamma_0}, \\ v = v_0, \quad \dot{v}(0) = v_1, \end{cases}$$

where

$$\hat{a}_{ij}(x,y) = a_{ij}(x, \nabla w + y),$$

$$\hat{b}(x,y) = \sum_{ij} a_{ij}(x, \nabla w + y) w_{x_i x_j} + b(x, \nabla w + y),$$

$$v_0 = w_0 - w, \quad v_1 = w_1.$$

Let $u \in \cap_{k=0}^m C^k\left([0,T], H^{m-k}(\Omega)\right)$ be a solution of the problem (1.1) for some $T > 0$. Suppose that

$$\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-1/2}(\Gamma_0)\right)$$

and

$$\varphi^{(k)} \in H^1\left((0,T) \times \Gamma_0\right), \quad 0 \le k \le m-1.$$

We introduce

$$\mathcal{E}(t) = \sum_{k=0}^{m} \|u^{(k)}(t)\|_{m-k}^2, \quad \mathcal{E}_\Gamma(t) = \sum_{k=0}^{m-2} \|\varphi^{(k)}(t)\|_{m-k-1/2,\Gamma_0}^2, \tag{2.2}$$

$$Q(t) = \sum_{k=1}^{m} \left(\|u^{(k)}(t)\|^2 + \|\nabla u^{(k-1)}(t)\|^2 \right), \tag{2.3}$$

$$Q_\Gamma(t) = \sum_{k=1}^{m} \left(\|\varphi^{(k)}(t)\|_{\Gamma_0}^2 + \|\nabla \varphi^{(k-1)}(t)\|_{\Gamma_0}^2 \right), \tag{2.4}$$

where $\|\cdot\|_0 = \|\cdot\|$ and $\|\cdot\|_\Gamma = \|\cdot\|_{0,\Gamma_0}$ are norms of $L^2(\Omega)$ and $L^2(\Gamma_0)$ and $\|\cdot\|_j$, $\|\cdot\|_{j,\Gamma_0}$ are norms of $H^j(\Omega)$, $H^j(\Gamma_0)$ for $1 \le j \le m$, respectively.

Theorem 2.1 *We consider solutions of the problem* (1.1) *near the zero equilibrium. Let $\gamma > 0$ be given and u be a solution of the problem* (1.1) *on the interval $[0, T]$ for some $T > 0$ such that*

$$\sup_{0 \le t \le T} \|u(t)\|_m \le \gamma. \tag{2.5}$$

Then there is $c_\gamma > 0$, which depends on the γ and but is independent of initial data (u_0, u_1) and boundary functions φ, such that

$$Q(t) \le \mathcal{E}(t) \le c_\gamma Q(t) + c_\gamma \mathcal{E}_\Gamma(t) + c_\gamma \sum_{k=2}^{m} \mathcal{E}^k(t), \quad 0 \le t \le T, \tag{2.6}$$

and

$$Q(t) \le c_\gamma Q(0) + c_\gamma \int_0^t \left[\left(1 + \mathcal{E}^{1/2}(t)\right) Q(t) + Q_\Gamma(t) + \sum_{k=2}^{m} \mathcal{E}^k(t) \right] dt, \tag{2.7}$$

for $t \in [0, T]$.

We collect here a few basic properties of Sobolev spaces to be invoked in the sequel.

(i) Let $s_1 > s_2 \ge 0$. For any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$\|w\|_{s_2}^2 \le \varepsilon \|w\|_{s_1}^2 + c_\varepsilon \|w\|^2 \quad \forall\ w \in H^{s_1}(\Omega). \tag{2.8}$$

(ii) If $s > n/2$, then for each $k = 0, \cdots$, we have $H^{s+k}(\Omega) \subset C^k(\overline{\Omega})$ with continuous inclusion.

(iii) If $r := \min\{s_1, s_2, s_1 + s_2 - [n/2] - 1\} \ge 0$, then there is a constant $c > 0$ such that

$$\|fg\|_r \le c\|f\|_{s_1} \|g\|_{s_2} \quad \forall\ f \in H^{s_1}(\Omega),\ g \in H^{s_2}(\Omega). \tag{2.9}$$

(iv) Let $s_j \ge 0$, $j = 1, \cdots, k$, and $r := \min_{1 \le i \le k} \min_{j_1 \le \cdots \le j_i} \{s_{j_1} + \cdots + s_{j_i} - (i-1)([n/2] + 1)\} \ge 0$. Then there is a constant $c > 0$ such that

$$\|f_1 \cdots f_k\|_r \le c\|f_1\|_{s_1} \cdots \|f_k\|_{s_k} \quad \forall\ f_j \in H^{s_j}(\Omega),\ 1 \le j \le k. \tag{2.10}$$

Let $u \in \cap_{k=0}^m C^k\left([0,T], H^{m-k}(\Omega)\right)$ be a solution of short time to the problem (1.1). We introduce a linear operator $B(t)$ by

$$B(t)w = -\sum_{ij=1}^n a_{ij}(x, \nabla u) w_{x_i x_j} \quad w \in H^1(\Omega).$$

Then

$$(B(t)w, v) = -\int_\Gamma v\langle A\nabla w, \nu\rangle d\Gamma + (A\nabla w, \nabla v) - (Cw, v), \quad w,\ v \in H^1(\Omega), \tag{2.11}$$

where $A = (a_{ij}(x, \nabla u))$, ν is the normal of Γ in the dot metric, and

$$Cw = -\sum_{ij=1}^n \left(a_{ij}(x, \nabla u)\right)_{x_j} w_{x_i}.$$

Then the problem (1.1) becomes

$$\begin{cases} \ddot{u}(t) + B(t)u(t) = b(x, \nabla u) \quad (t, x) \in (0, T) \times \Omega, \\ u(t)|_{\Gamma_1} = 0, \quad u(t)|_{\Gamma_0} = \varphi(t), \quad 0 \in (0, T), \\ u(0) = w_0, \quad \dot{u}(0) = w_1, \quad x \in \Omega. \end{cases} \tag{2.12}$$

Lemma 2.1 *(i) Let $f(x, y)$ be a smooth function on $\overline{\Omega} \times \mathcal{R}^n$. Set $F(x) = f(x, \nabla u)$. For $0 \le k \le m-1$, there is $c = c(\sup_{x\in\Omega} |\nabla u|) > 0$ such that*

$$\|F\|_k \le c \sum_{j=0}^k (1 + \|u\|_m)^j. \tag{2.13}$$

(ii) Let u be a solution of the problem (1.1) and $\gamma > 0$ be given. Suppose that the condition (2.5) holds true. Then there is $c_\gamma > 0$, which depends on the γ, such that

$$\|v\|_{k+1}^2 \le c_\gamma \left(\|B(t)v\|_{k-1}^2 + \|v\|_{k+1/2,\,\Gamma}^2 + \|v\|_k^2\right), \quad v \in H^k(\Omega), \tag{2.14}$$

for $0 \le k \le m-1$.

Proof. (i) By induction. The inequality (2.13) is clearly true for $k = 0$. Suppose that it holds for $0 \le k < m-1$. Since

$$F_{x_i} = f_{x_i}(x, \nabla u) + \sum_{j=1}^n f_{y_j}(x, \nabla u) u_{x_i x_j}, \quad 1 \le i \le n,$$

by using the formula (2.9) and the induction assumption for $f_{x_i}(x, \nabla u)$ and for $f_{y_j}(x, \nabla u)$, respectively, we obtain

$$\|F\|_{k+1} = (\|F\|^2 + \sum_{i=1}^n \|F_{x_i}\|_k^2)^{1/2}$$

$$\le c + c\sum_{i=1}^{n} \|f_{x_i}(x,\nabla u)\|_k + c\sum_{ij=1}^{n} \|f_{y_j}(x,\nabla u)\|_k \|u_{x_i x_j}\|_{m-2}$$

$$\le c + c\sum_{j=0}^{k}(1+\|u\|_m)^j + c\sum_{j=0}^{k}(1+\|u\|_m)^j\|u\|_m$$

$$\le c\sum_{j=0}^{k+1}(1+\|u\|_m)^j.$$

(ii) A standard method as to the linearly elliptic problem can give the inequality (2.14), for example see Taylor [22].

Lemma 2.2 *Let $\gamma > 0$ be given and u be a solution of the problem* (1.1) *on the interval* $[0,T]$ *for some $T > 0$ such that the condition* (2.5) *holds true. Then there is $c_\gamma > 0$, which depends on the γ, such that*

$$\|b^{(k)}(x,\nabla u)\|^2_{m-k-2} \le c_\gamma \|u^{(k)}(t)\|^2_{m-k-1} + c_\gamma \sum_{i=2}^{k} \mathcal{E}^i(t), \quad 1 \le k \le m-1, \tag{2.15}$$

and

$$\|B^{(j)}(t)u^{(k-j)}(t)\|^2_{m-k-2} \le c_\gamma \sum_{i=1}^{k} \mathcal{E}^{1+i}(t), \quad 1 \le j \le k \le m-1. \tag{2.16}$$

Proof. We have

$$b^{(k)}(x,\nabla u) = \sum_{i=1}^{k} \sum_{r_1+\cdots+r_i=k} D^i_y b\left(\nabla u^{(r_1)}(t),\cdots,\nabla u^{(r_i)}(t)\right), \tag{2.17}$$

where $D^i_y b$ denotes the covariant differential of i order of the function $b(x,y)$ with respect to the variable y in the dot metric of $\mathcal{R}^n$.

Let us see the term $D_y b(\nabla u^{(k)})$ first. We have

$$D_y b(\nabla u^{(k)}) = \sum_{l=1}^{n} b_{y_l}(x,\nabla u) u^{(k)}_{x_l}(t).$$

By (2.9) and (2.13)

$$\|b_{y_l} u^{(k)}_{x_l}\|_{m-k-2} \le c\|b_{y_l}\|_{m-1}\|u^{(k)}(t)\|_{m-k-1} \le c_\gamma \|u^{(k)}(t)\|_{m-k-1}. \tag{2.18}$$

Let $2 \le i \le k$. We observe that $D^i_y b\left(\nabla u^{(r_1)}(t),\cdots,\nabla u^{(r_i)}(t)\right)$ is a sum of terms such as

$$f(x,\nabla u) u^{(r_1)}_{x_{l_1}}(t)\cdots u^{(r_i)}_{x_{l_i}}(t)$$

where $r_1 + \cdots + r_i = k$. Using (2.10) and (2.13), we have

$$\|f u^{(r_1)}_{x_{l_1}} \cdots u^{(r_i)}_{x_{l_i}}\|_{m-k-2} \le c_\gamma \|u^{(r_1)}(t)\|_{m-r_1} \cdots \|u^{(r_i)}(t)\|_{m-r_i} \le c_\gamma \mathcal{E}^{i/2}(t). \tag{2.19}$$

(2.15) follows from (2.18) and (2.19). A similar argument yields (2.16).

Lemma 2.3 *Let $\gamma > 0$ be given and u be a solution of the problem* (1.1) *on the interval* $[0, T]$ *for some $T > 0$ such that the condition* (2.5) *holds true. Then there is $c_\gamma > 0$ such that the inequality* (2.6) *is true.*

Proof. It is clear that

$$\|u^{(m)}(t)\|^2 + \|u^{(m-1)}(t)\|_1^2 = \|u^{(m)}(t)\|^2 + \|\nabla u^{(m-1)}(t)\|^2 + \|u^{(m-1)}(t)\|^2 \le \mathcal{Q}(t). \quad (2.20)$$

Proceeding by induction, we assume that for some $1 \le j \le m$

$$\|u^{(j)}(t)\|_{m-j}^2 \le c_\gamma Q(t) + c_\gamma \mathcal{E}_\Gamma(t) + c_\gamma \sum_{k=2}^{m} \mathcal{E}^k(t), \quad (2.21)$$

which, as shown above, is true for $j = m$ and $j = m - 1$. Formal differentiation of the equation in (2.12) $j - 2$ times with respect to t yields

$$u^{(j)}(t) + B(t)u^{(j-2)}(t) = b^{(j-2)}(x, \nabla u) - \sum_{k=1}^{j-2} C_k B^{(k)}(t) u^{(j-2-k)}(t). \quad (2.22)$$

Using Lemmas 2.1, 2.2, (2.22), and (2.8), we obtain

$$\begin{aligned} \|u^{(j-2)}(t)\|_{m-j+2}^2 &\le c_\gamma \|B(t)u^{(j-2)}(t)\|_{m-j}^2 + c_\gamma \|\varphi^{(j-2)}(t)\|_{m-j+3/2,\,\Gamma_0}^2 + c_\gamma \|u^{(j-2)}(t)\|_{m-j+1}^2 \\ &\le c_\gamma \|u^{(j)}(t)\|_{m-j}^2 + c_\gamma \|u^{(j-2)}(t)\|_{m-j+1}^2 + c_\gamma \mathcal{E}_\Gamma(t) + c_\gamma \sum_{i=2}^{j-1} \mathcal{E}^i(t). \quad (2.23) \end{aligned}$$

The inequality (2.6) follows by induction by using the following inequality in (2.23)

$$\|u^{(j-2)}(t)\|_{m-k+1}^2 \le \varepsilon \|u^{(j-2)}(t)\|_{m-j+2}^2 + c_{\gamma,\varepsilon} \|u^{(j-2)}(t)\|^2.$$

Lemma 2.4 *Let $\gamma > 0$ be given and u be a solution of the problem* (1.1) *on the interval* $[0, T]$ *for some $T > 0$ such that the condition* (2.5) *holds true. Let $w \in H^1((0, T) \times \Omega)$ solve the linear problem*

$$\begin{cases} \ddot{w}(t) + B(t)w(t) = F(t) & (t, x) \in (0, T) \times \Omega, \\ w|_{\Gamma_1} = 0, \quad w|_{\Gamma_0} = \varphi & (t, x) \in (0, T) \times \Gamma, \\ w(0) = w^0, \quad \dot{w}(0) = w^1, & x \in \Omega. \end{cases} \quad (2.24)$$

Set

$$\Upsilon(t) = \|\dot{w}(t)\|^2 + \|\nabla w(t)\|^2, \quad \Upsilon_\Gamma(t) = \|\dot{\varphi}(t)\|_{\Gamma_0}^2 + \|\nabla_\Gamma \varphi\|_{\Gamma_0}^2,$$

where ∇_Γ is the gradient of Γ in the induced metric by the dot metric of $\mathcal{R}^n$. Then there is $c_\gamma > 0$ such that

$$\Upsilon(t) \le c_\gamma \Upsilon(0) + c_\gamma \int_0^t \left[(1 + \|\dot{u}(t)\|_{m-1}) \Upsilon(\tau) + \Upsilon_\Gamma(\tau) + \|F(\tau)\|^2 \right] d\tau \quad 0 \le t \le T. \quad (2.25)$$

Proof. Let

$$P(t) = \|\dot{w}(t)\|^2 + (A\nabla w, \nabla w)\,. \tag{2.26}$$

Using (2.11) and (2.46), we obtain

$$\begin{aligned}\dot{P}(t) &= 2\left(\ddot{w}(t), \dot{w}(t)\right) + 2\left(A\nabla w(t), \nabla\dot{w}(t)\right) + \left(\dot{A}\nabla w, \nabla w\right)\\ &= 2\left(F + Cw,\ \dot{w}\right) + \left(\dot{A}\nabla w, \nabla w\right) + 2\int_{\Gamma_0} \dot{\varphi} w_{\nu_A} d\Gamma,\end{aligned} \tag{2.27}$$

where $w_{\nu_A} = \langle A(x, \nabla u)\nabla w,\ \nu\rangle$. It follows from (2.27) that

$$\begin{aligned}\Upsilon(t) &\le c_\gamma P(t) \le c_\gamma P(0) + c_\gamma \int_0^t \left[(\|\nabla w\| + \|F(t)\|)\|\dot{w}\| + \|\dot{u}\|_{m-1}\|\nabla w\|^2\right] dt\\ &+\varepsilon \int_0^t \int_{\Gamma_0} w_{\nu_A}^2 d\Gamma dt + c_{\gamma,\varepsilon} \int_0^t \int_{\Gamma_0} \dot{\varphi}^2 d\Gamma dt, \quad 0 \le t \le T,\end{aligned} \tag{2.28}$$

where $\varepsilon > 0$ will be determined later.

To obtain (2.25) from (2.28), we have to estimate the term $\int_0^t \int_\Gamma w_{\nu_A}^2 d\Gamma dt$.

We now introduce a Riemannian metric

$$g = A^{-1}(x, \nabla u)$$

on $\overline{\Omega}$ so that the couple $(\overline{\Omega}, g)$ is a Riemannian manifold. Let H be a vector field on $\overline{\Omega}$ such that

$$H|_{\Gamma_1} = 0, \quad \text{in a neigborhood of } \Gamma_1; \quad H|_{\Gamma_0} = \nu_A. \tag{2.29}$$

We have the following formula (see Yao [24], Lemma 2.1)

$$\begin{aligned}\langle A\nabla w, \nabla\left(H(w)\right)\rangle &= D_g H\left(\nabla_g w, \nabla_g w\right) + \frac{1}{2}\operatorname{div}\left(|\nabla_g w|_g^2 H\right)\\ &-\frac{1}{2}|\nabla_g w|_g^2 \operatorname{div} H,\end{aligned} \tag{2.30}$$

where $D_g H$ is the covariant differential of the vector field H and $\nabla_g = A(x, \nabla u)\nabla$ is the gradient of the Riemmannian metric g.

We multiply the two sides of the equation in (2.24) by $H(w)$ and integrate over Ω by parts, via the formulas (2.11), (2.29), and (2.30) to obtain

$$\begin{aligned}&\int_{\Gamma_0} \left[w_{\nu_A}^2 + \frac{1}{2}(\dot{\varphi}^2 - |\nabla_g w|_g^2)|\nu_A|_g^2\right] d\Gamma\\ &= \frac{\partial}{\partial t}\left(\dot{w}, H(w)\right) + \int_\Omega \left[D_g H(\nabla_g w, \nabla_g w) + \frac{1}{2}(\dot{w}^2 - |\nabla_g w|_g^2)\operatorname{div} H\right] dx\\ &-\int_\Omega \left(Cw + F(t)\right) H(w) dx.\end{aligned} \tag{2.31}$$

Using the formula (2.31) and the relation

$$|\nabla_g w|_g^2 = \frac{w_{\nu_A}^2}{|\nu_A|_g^2} + |\nabla_{\Gamma_g}\varphi|_g^2, \quad x \in \Gamma,$$

where ∇_{Γ_g} is the gradient of Γ in the induced metric by the Riemannian metric g, we have

$$\int_0^t \int_{\Gamma_0} w_{\nu_A}^2 d\Gamma dt \leq c_\gamma \left[\Upsilon(t) + \Upsilon(0)\right] + c_\gamma \int_0^t \int_\Omega \left[\Upsilon(t) + \Upsilon_\Gamma(t)\right] dxdt. \tag{2.32}$$

Finally, we insert the inequality (2.32) into the inequality (2.28), choose a $\varepsilon > 0$ so small such that the term $\varepsilon c_\gamma \Upsilon(t)$ can be moved to the left hand side of the inequality to obtain the inequality (2.25). $\|$

The Proof of Theorem 2.1 Lemma 2.3 gives the inequality (2.6). Let us prove the inequality (2.7).

We take $w = u^{(j-2)}(t)$ for $2 \leq j \leq m+1$ from the equation (2.22) and apply Lemma 2.4 to obtain

$$\begin{aligned}
&\|u^{(j-1)}(t)\|^2 + \|\nabla u^{(j-2)}(t)\|^2 \\
&\leq c_\gamma Q(0) + c_\gamma \int_0^t \left[\left(1 + \mathcal{E}^{1/2}(t)\right) Q(t) + Q_\Gamma(t)\right] dt \\
&\quad c_\gamma \int_0^t \left(\|b^{(j-2)}\|^2 + \sum_{k=1}^{j-2} \|B^{(k)}(t) u^{(j-2-k)}(t)\|^2\right) dt.
\end{aligned} \tag{2.33}$$

In addition, a similar computation as in Lemma 2.2 yields

$$\begin{aligned}
\|b^{(j-2)}\|^2 &\leq c_\gamma \|\nabla u^{(j-2)}\|^2 + c_\gamma \sum_{k=2}^m \mathcal{E}^k(t) \\
&\leq c_\gamma Q(t) + c_\gamma \sum_{k=2}^m \mathcal{E}^k(t),
\end{aligned} \tag{2.34}$$

and

$$\|B^{(k)}(t) u^{(j-2-k)}(t)\|^2 \leq c_\gamma \sum_{k=2}^m \mathcal{E}^k(t), \quad 1 \leq k \leq j-2. \tag{2.35}$$

The inequality (2.7) follows from (2.33)-(2.35). $\|$

The Proof of Theorem 1.1 Clearly, it will suffice to prove Theorem 1.1 for the zero equilibrium $w = 0$.

Let $T_1 > 0$ be arbitrary given. We take $\gamma = 1$. Let

$$c_1 = c_\gamma \geq 1 \tag{2.36}$$

be fixed such that the corresponding inequalities (2.6) and (2.7) of Theorem 2.1 hold for t in the existence interval of the solution u, respectively.

We shall prove that, if initial data (w_0, w_1) and boundary value φ are compatible of m order to satisfy

$$\mathcal{E}(0) + \max_{0 \leq t \leq T_1} \mathcal{E}_\Gamma(t) + \int_0^{T_1} Q_\Gamma(t) dt \leq \frac{1}{16 c_1^3} e^{-4 c_1^2 T_1}, \tag{2.37}$$

then the solution of the problem (1.1) exists at least on the interval $[0, T_1]$.

We set

$$\eta = \frac{1}{4c_1} \leq \frac{1}{4} < \frac{1}{2}. \tag{2.38}$$

Since $\mathcal{E}(0) \leq \eta/4$, the solution of short time must satisfy

$$\mathcal{E}(t) \leq \eta \leq 1/2 \tag{2.39}$$

for some interval $[0, \delta]$.

Let δ_0 be the largest number such that (2.39) is true for $t \in [0, \delta_0)$. We shall prove $\delta_0 \geq T_1$ by contradiction.

Suppose that $\delta_0 < T_1$. In this interval $[0, \delta_0]$ the condition (2.5) is true, we apply Theorem 2.1, and the inequalities (2.6) and (2.7), via (2.36), (2.38), and (2.39), imply

$$\mathcal{E}(t) \leq 2c_1^2 \left[\mathcal{E}(0) + \max_{0 \leq t \leq T_1} \mathcal{E}_\Gamma(t) + \int_0^{T_1} Q_\Gamma(t) dt \right] + 4c_1^2 \int_0^t \mathcal{E}(t) dt, \tag{2.40}$$

for $t \in [0, \delta_0]$. By (2.37) and (2.40), the Gronwall inequality yields

$$\mathcal{E}(\delta_0) \leq \eta/2 < \eta.$$

This is a contradiction. $\|$

Proof Theorem 1.4 This proof follows by an similar argument in the proof of Theorem 1.1 which this time is based on the estimates of the following theorem.

We turn to the problem (1.18) with the Neumann data on the portion Γ_0 of the boundary. Let $u \in \cap_{k=0}^m C^k \left([0, T], H^{m-k}(\Omega) \right)$ be a solution of the problem (1.18) for some $T > 0$. We introduce an operator

$$\mathcal{B}(t) v = \operatorname{div} B \nabla v \quad v \in H^2(\Omega), \tag{2.41}$$

where

$$B = \left(b_{ij}(x, \nabla u) \right), \quad b_{ij}(x, y) = \int_0^1 a_{iy_j}(x, \sigma y) d\sigma.$$

Then

$$\operatorname{div} \mathbf{a}(x, \nabla u) = \operatorname{div} B \nabla u, \quad \varphi = \langle \mathbf{a}(x, \nabla u), \nu \rangle = \langle B(x, \nabla u) \nabla u, \nu \rangle = u_{\upsilon_B}. \tag{2.42}$$

Suppose that

$$\varphi \in \cap_{k=0}^{m-2} C^k \left([0, T], H^{m-k-3/2}(\Gamma_0) \right). \tag{2.43}$$

We introduce

$$\mathcal{E}_{\Gamma N}(t) = \sum_{k=0}^{m-2} \| \varphi^{(k)}(t) \|^2_{m-k-3/2, \Gamma_0},$$

$$Q_{\Gamma N} = \sum_{k=0}^{m-1} \|\varphi^{(k)}\|^2_{1/2,\Gamma_0} dt.$$

Then

Theorem 2.2 *Let $\gamma > 0$ be given and u be a solution of the problem* (1.18) *on the interval $[0,T]$ for some $T > 0$ such that the inequality* (2.5) *is true. Then there is $c_\gamma > 0$, which only depends on the γ, such that*

$$Q(t) \leq \mathcal{E}(t) \leq c_\gamma Q(t) + c_\gamma \mathcal{E}_{\Gamma N}(t) + c_\gamma \sum_{k=2}^{m} \mathcal{E}^k(t), \quad 0 \leq t \leq T, \tag{2.44}$$

and

$$Q(t) \leq c_\gamma Q(0) + c_\gamma \int_0^t \left[\left(1 + \mathcal{E}^{1/2}(t)\right) Q(t) + Q_{\Gamma N}(t) + \sum_{k=2}^{m} \mathcal{E}^k(t) \right] dt, \tag{2.45}$$

for $t \in [0,T]$, where $\mathcal{E}(t)$ and $Q(t)$ are given in (2.2) *and* (2.3)*, respectively.*

Proof. It will suffice to make some revisions on the proofs of Lemmas 2.3 and 2.4, respectively.

Using the ellipticity that there is $c_\gamma > 0$ such that

$$\|w\|^2_{k+1} \leq c_\gamma \left(\|\mathcal{B}(t)w\|^2_{k-1} + \|w_{\nu_B}\|^2_{k-1/2,\,\Gamma} + \|w\|^2_k \right), \quad w \in H^k(\Omega),$$

for $0 \leq k \leq m-1$, in the proof of Lemma 2.3 yields the inequality (2.44).

Moreover, the second inequality (2.45) is based on the following

Lemma 2.5 *Let $\gamma > 0$ be given and u be a solution of the problem* (1.18) *on the interval $[0,T]$ for some $T > 0$ such that the condition* (2.5) *holds true. Let $w \in H^1((0,T) \times \Omega)$ solve the linear problem*

$$\begin{cases} \ddot{w}(t) = \mathcal{B}(t)w + F(t) & (t,x) \in (0,T) \times \Omega, \\ w_{\Gamma_1} = 0, \quad w_{\nu_B}|_{\Gamma_0} = \varphi & (t,x) \in (0,T) \times \Gamma, \\ w(0) = w^0, \quad \dot{w}(0) = w^1, \quad x \in \Omega, \end{cases} \tag{2.46}$$

where

$$w_{\nu_B} = \langle B\nabla w, \nu \rangle. \tag{2.47}$$

Set

$$\Upsilon(t) = \|\dot{w}(t)\|^2 + \|\nabla w(t)\|^2, \quad \Upsilon_{\Gamma N}(t) = \|\varphi(t)\|^2_{H^{1/2}(\Gamma_0)}.$$

Then there is $c_\gamma > 0$ such that

$$\Upsilon(t) \leq c_\gamma \Upsilon(0) + c_\gamma \int_0^t \left[(1 + \|\dot{u}(t)\|_{m-1}) \Upsilon(\tau) + \Upsilon_{\Gamma N}(\tau) + \|F(\tau)\|^2 \right] d\tau, \tag{2.48}$$

for $0 \leq t \leq T$.

Proof. Let

$$P(t) = \|\dot{w}\|^2 + (B\nabla w, \nabla w)\,.$$

Then

$$\dot{P}(t) = 2\,(F,\ \dot{w}) + \left(\dot{B}\nabla w, \nabla w\right) + 2\int_{\Gamma_0} \varphi\dot{w}\Gamma. \tag{2.49}$$

Using the estimate

$$|(\varphi, \dot{w})_{L^2(\Gamma_0)}| \le \|\dot{w}\|_{H^{-1/2}(\Gamma_0)}\|\varphi\|_{H^{1/2}(\Gamma_0)} \le c\Upsilon(t) + c\Upsilon_{\Gamma N}(t)$$

in (2.49) gives the inequality (2.48).

3 Locally exact controllability; the Dirichlet action

The first step of the proof for the local exact controllability depends on the following fact: Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and Φ: $\mathcal{O} \to \mathcal{Y}$, where $\mathcal{O}$ is an open subset of $\mathcal{X}$, be Frechét differentiable. If $\Phi'(X_0)$: $\mathcal{X} \to \mathcal{Y}$ is surjective, then there is an open neighbourhood of $Y_0 = \Phi(X_0)$ contained in the image $\Phi(\mathcal{O})$.

We start by specifying a value of T about which we shall say more later. We introduce a Banach space $\mathcal{X}_0^m(T)$ as follows. $\mathcal{X}_0^m(T)$ consists of all the functions

$$\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-1/2-k}(\Gamma_0)\right), \quad \varphi^{(k)} \in H^1\left((0,T)\times\Gamma_0\right), \tag{3.1}$$

$$\varphi^{(k)}(0) = 0, \quad x \in \Gamma_0, \quad 0 \le k \le m-1, \tag{3.2}$$

with the norm

$$\|\varphi\|^2_{\mathcal{X}_0^m(T)} = \sum_{k=0}^{m-2} \|\varphi^{(k)}\|^2_{C\left([0,T],H^{m-k-3/2}(\Gamma_0)\right)} + \sum_{k=0}^{m-1} \|\varphi^{(k)}\|^2_{H^1((0,T)\times\Gamma_0)}. \tag{3.3}$$

Let an equilibrium solution $w \in H^m_{\Gamma_1}(\Omega)$ be given. We invoke Theorem 1.1 to define a map for $\varphi \in \mathcal{X}_0^m(T)$ by setting

$$\Phi(\varphi) = (u(T), \dot{u}(T))\,, \tag{3.4}$$

where u is the solution of the following problem

$$\begin{cases} \ddot{u} = \sum a_{ij}(x,\nabla u)u_{x_ix_j} + b(x,\nabla u) \quad (t,x) \in (0,T)\times\Omega, \\ u|_{\Gamma_1} = 0, \quad t \in (0,T), \\ u|_{\Gamma_0} = w|_{\Gamma_0} + \varphi, \quad t \in (0,T), \\ u(0) = w, \quad \dot{u}(0) = 0. \end{cases} \tag{3.5}$$

Let $\varepsilon_T > 0$ be given by Theorem 1.1. Then

$$\Phi: \quad B_{\mathcal{X}_0^m(T)}(0,\varepsilon_T) \to \left(H^m(\Omega)\cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^{m-1}(\Omega)\cap H^1_{\Gamma_1}(\Omega)\right), \tag{3.6}$$

where $B_{\mathcal{X}_0^m(T)}(0,\varepsilon_T) \subset \mathcal{X}_0^m(T)$ is the ball with the radius ε_T centered at 0. We observe that $\Phi(0) = (w, 0)$.

We need to evaluate

$$\Phi'(0)\varphi = \frac{\partial}{\partial\sigma}\Phi(\sigma\varphi)|_{\sigma=0}, \quad \varphi \in \mathcal{X}_0^m(T). \tag{3.7}$$

It is easy to check that

$$\Phi'(0)\varphi = (v(T), \dot{v}(T)), \tag{3.8}$$

where $v(t,x)$ is the solution of the linear system with variable coefficients in the space variable

$$\begin{cases} \ddot{v} = \mathcal{A}v + F(v), \quad (t,x) \in (0,T)\times\Omega, \\ v|_{\Gamma_1} = 0, \quad t \in (0,T), \\ v|_{\Gamma_0} = \varphi, \quad t \in (0,T), \\ v(0) = \dot{v}(0) = 0, \end{cases} \tag{3.9}$$

where

$$\mathcal{A}v = \sum_{ij=1}^{n} a_{ij}(x,\nabla w)v_{x_ix_j}, \tag{3.10}$$

$$F = (F_1, \cdots, F_n), \tag{3.11}$$

$$F_i = \sum_{lj}[a_{ljy_i}(x,\nabla w)(w_{x_lx_j} - a_{ijy_l}(x,\nabla w)w_{x_lx_j}] + b_{y_i}(x,\nabla w).$$

We now verify that $\Phi'(0)$ is surjective. In the language of control theory surjection is just exact controllability, which for a reversible system such as (3.9) is equivalent to null controllability.

Explicitly, one has to show that, for specified T, given $v^0 \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ and $v^1 \in H^{m-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$, one can find $\varphi \in \tilde{\mathcal{X}}_0^m(T)$ such that the solution to

$$\begin{cases} \ddot{v} = \mathcal{A}v + F(v), \quad (t,x) \in (0,T)\times\Omega, \\ v|_{\Gamma_1} = 0, \quad t \in (0,T), \\ v|_{\Gamma_0} = \varphi, \quad t \in (0,T), \\ v(0) = v^0, \quad \dot{v}(0) = v^1 \end{cases} \tag{3.12}$$

satisfies

$$v(T) = \dot{v}(T) = 0, \tag{3.13}$$

where $\tilde{\mathcal{X}}_0^m(T)$ is the Banach space of all function with (3.1) and the norm (3.3) but with (3.2) replaced by

$$\varphi^{(k)}(T) = 0, \quad x \in \Gamma_0, \quad 0 \le k \le m-1. \tag{3.14}$$

Theorem 1.2 is then established by the following

Theorem 3.1 *Let an equilibrium $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable. Let $T > T_0$ be given where T_0 is defined by* (1.13)*. Then, for any*

$$(v^0, v^1) \in \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^{m-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right),$$

there is a $\varphi \in \tilde{\mathcal{X}}^m_0(T)$ such that the solution

$$v \in \cap_{k=0}^m C^k\left([0,T], H^{m-k}(\Omega)\right)$$

of the problem (3.12) *satisfies* (3.13)*.*

Distributed Control. As to the exact controllability of linear systems by distributed control there is a long history and the results are rich where many approaches are involved. Here the distributed control means that solutions $(v(t), \dot{v}(t))$ of the controlled system (3.12) are only in the space $L^2(\Omega) \times H^{-1}(\Omega)$ for $t \in [0, T]$. We just mention what we need in this paper. One of the useful approaches is the multiplier method below, introduced by Ho [12] and Lions [17], to control the linear system by its duality system.

We start with the wave equation

$$\begin{cases} \ddot{\phi} = \mathcal{B}\phi, & (t,x) \in (0,T) \times \Omega, \\ \phi|_\Gamma = 0, & t \in (0,T), \\ \phi(0) = \phi_0, & \dot{\phi}(0) = \phi_1, \end{cases} \tag{3.15}$$

where the operator $\mathcal{B}$ is defined by

$$\mathcal{B}v = \mathcal{A}v - F(v) - v \operatorname{div} F, \quad v \in H^2(\Omega), \tag{3.16}$$

that is the dual system of the system (3.12). We have the following Green formula

$$(v, \mathcal{B}u) = (\mathcal{B}^\star v,\, u) + \int_\Gamma [v u_{\nu_A} - u v_{\nu_A} - uv\langle AF, \nu\rangle]\, d\Gamma, \quad u,\, v \in H^2(\Omega), \tag{3.17}$$

where

$$\mathcal{B}^\star v = \mathcal{A}v + F(v). \tag{3.18}$$

Given $(\phi_0, \phi_1) \in H^1_0(\Omega) \times L^2(\Omega)$, the problem (3.15) admits a unique solution. We then solve the problem

$$\begin{cases} \ddot{\psi} = \mathcal{A}\psi + F(\psi), & (t,x) \in (0,T) \times \Omega, \\ \psi(T) = \dot{\psi}(T) = 0, & x \in \Omega, \\ \psi|_{\Gamma_1} = 0, \quad \psi|_{\Gamma_0} = \phi_{\nu_A}, & t \in (0,T), \end{cases} \tag{3.19}$$

where $\phi_{\nu_A} = \langle A\nabla\phi, \nu\rangle$, $A = (a_{ij}(x, \nabla w))$, and ϕ is produced by (3.15). Let ψ be the solution of the problem (3.19). We then have constructed a control ϕ_{ν_A} on $(0,T) \times \Gamma_0$ moving the initial state $\left(\psi(0), \dot{\psi}(0)\right)$ to rest at the time T.

We define a mapping Λ: $H_0^1(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$ by

$$\Lambda(\phi_0, \phi_1) = (\dot{\psi}(0), -\psi(0)). \tag{3.20}$$

A formal use of Green's formula yields, after we multiply (3.15) by ϕ and integrate by parts over $\Sigma = (0, T) \times \Omega$,

$$\langle \Lambda(\phi_0, \phi_1), (\phi_0, \phi_1) \rangle_{L^2(\Omega) \times L^2(\Omega)} = \int_{\wp_0} \phi_{\nu_A}^2 \wp, \tag{3.21}$$

where $\wp_0 = (0, T) \times \Gamma_0$.

Let constants $c_1 > 0$ and $c_2 > 0$ be such that

$$c_1 E(\phi_0, \phi_1) \le \int_{\wp_0} \phi_{\nu_A}^2 d\wp \le c_2 E(\phi_0, \phi_1), \tag{3.22}$$

for $(\phi_0, \phi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ where

$$E(\phi_0, \phi_1) = \|A^{1/2} \nabla \phi_0\|^2 + \|\phi_1\|^2. \tag{3.23}$$

Then, for any $(\phi_0, \phi_1) \in H_0^1(\Omega) \times H^1(\Omega)$, one has $\phi_{\nu_A} \in L^2\left((0, T) \times \Gamma_0\right)$ that drives the system starting from $(\psi(0), \dot{\psi}(0))$ at the time $t = 0$ to rest at the time T.

Then the key point is to establish the inequality (3.22). For $\mathcal{A}$ being the classical Laplacian and $F = 0$, the inequality (3.22) was proved in Ho [12]. For $\mathcal{A}$ with variable coefficients in space, such as (3.10), and $F = 0$, the inequality (3.22) was established under some geometric conditions in Yao [24], where the geometrical method was introduced. Without geometric conditions, the inequality (3.22) is not true even if the control portion Γ_0 of Γ is the whole boundary. A counterexample was given by Yao [24]. Then the geometrical method was extended by Lasiecka, Triggiani, and Yao [15] to include the case of the first order terms $F \neq 0$. This method was again extended to study the modeling and control problems of thin shells by Chai etc., [5], [6], and Lasiecka, etc., [14]. A recent survey paper on the geometrical method is by Gulliver, etc., [10].

The lemma below follows by Lasiecka, Triggiani, and Yao [15], Theorem 3.2, where a uniqueness result, needed, is provided by Triggiani and Yao [23], Theorem 10.1.1.

Lemma 3.1 *Let an equilibrium $w \in H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable and T_0 be given by the formula* (1.13). *Then, for $T > T_0$ given, Λ is an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$. In particular, there are $c_1 > 0$ and $c_2 > 0$ such that the inequality* (3.22) *holds true.*

However, the above control strategy only gives distributed control functions because solutions $(\psi(t), \dot{\psi}(t))$ of the controlled system (3.19) are only in $L^2(\Omega) \times H^{-1}(\Omega)$ no matter (ϕ_0, ϕ_1) are smooth or not. Indeed, since $\phi_{\nu_A}(T) \neq 0$ for any $x \in \Gamma_0$, the compatible

condition $\psi(T) = \phi_\nu(T)$ for $x \in \Gamma_0$ is never true.

Smooth Control. We shall modify the above control strategy to obtain smooth controls to meet the need of Theorem 3.1.

Let $k \geq 1$ be an integer. Let $\Xi_0^k(\Omega)$ consist of the functions u in $H^k(\Omega)$ with the boundary conditions

$$\begin{cases} \mathcal{B}^i u|_{\Gamma_0} = 0, \quad 0 \leq i \leq l-1, \quad \text{if} \quad k = 2l; \\ \mathcal{B}^i u|_{\Gamma_0} = 0, \quad 0 \leq i \leq l \quad \text{if} \quad k = 2l+1, \end{cases} \tag{3.24}$$

and with the norms of $H^k(\Omega)$ where $\mathcal{B}$ is given by (3.16).

Let T_0 be given by the formula (1.13) and $T > T_1 > T_0$ be given. We assume that $z \in C^\infty(-\infty, \infty)$ is such that $0 \leq z(t) \leq 1$ with

$$z(t) = \begin{cases} 0, & t \geq T, \\ 1, & t \leq T_1. \end{cases} \tag{3.25}$$

For $(\phi_0, \phi_1) \in \Xi_0^{m+1}(\Omega) \times \Xi_0^m(\Omega)$ given, we solve the problem (3.15) and then, in stead of (3.19), we solve the following problem

$$\begin{cases} \ddot{\psi} = \mathcal{A}\psi + F(\psi), \quad (t,x) \in (0,T) \times \Omega, \\ \psi(T) = \dot{\psi}(T) = 0, \quad x \in \Omega, \\ \psi|_{\Gamma_1} = 0, \quad \psi|_{\Gamma_0} = z\phi_{\nu_A}, \quad t \in (0,T). \end{cases} \tag{3.26}$$

Let Λ be given by (3.20) where ψ in (3.20) are solutions of the problem (3.26) this time. It is easy to check that, for any (ϕ_0, ϕ_1), $(\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\langle \Lambda(\phi_0, \phi_1), (\varphi_0, \varphi_1) \rangle_{L^2(\Omega) \times L^2(\Omega)} = \int_{\wp_0} z(t) \phi_{\nu_A} \varphi_{\nu_A} d\wp, \tag{3.27}$$

with $\wp_0 = (0,T) \times \Gamma_0$, where ϕ and φ are solutions of the problem (3.15) with initial data (ϕ_0, ϕ_1) and (φ_0, φ_1), respectively.

We shall show that the problem (3.26) provides smooth controls to Theorem 3.1 by the following lemma.

Lemma 3.2 *Let $k \geq 0$ be an integer and Λ be given by* (3.20) *where ψ is the solution of the problem* (3.26). *There are then $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \|(\phi_0, \phi_1)\|_{H^{k+1}(\Omega) \times H^k(\Omega)} \leq \|\Lambda(\phi_0, \phi_1)\|_{H^{k-1}(\Omega) \times H^k(\Omega)} \leq c_2 \|(\phi_0, \phi_1)\|_{H^{k+1}(\Omega) \times H^k(\Omega)},$$
$$\forall \ (\phi_0, \phi_1) \in \Xi_0^{k+1}(\Omega) \times \Xi_0^k(\Omega). \tag{3.28}$$

In particular, Λ are isomorphisms from $\Xi_0^2(\Omega) \times \Xi_0^1(\Omega)$ onto $L^2(\Omega) \times H_{\Gamma_1}^1(\Omega)$ and from $\Xi_0^{k+1}(\Omega) \times \Xi_0^k(\Omega)$ onto $\left(H^{k-1}(\Omega) \cap H_{\Gamma_1}^1(\Omega)\right) \times \left(H^k(\Omega) \cap H_{\Gamma_1}^1(\Omega)\right)$ for $k \geq 2$, respectively.

Proof. Lemma 3.1 shows that the inequality (3.28) is true for $k = 0$.

We now proceed to prove the inequality (3.28) by induction on k. Let the inequality (3.28) be true for some integer $k \geq 0$. We want to show that the inequality (3.28) hold with k replaced by $k+1$.

Case I Let $k = 2l$ for some $l \geq 1$.

Let

$$(\phi_0, \phi_1) \in \Xi_0^{k+2}(\Omega) \times \Xi_0^{k+1}(\Omega) \tag{3.29}$$

be given. Suppose that ϕ is the solution of the problem (3.15) corresponding to the initial data (ϕ_0, ϕ_1). Then $\phi^{(2i)}$ and $\phi^{(2i+1)}$ are the solutions of the problem (3.15) corresponding to the initial data $(\mathcal{B}^i\phi_0, \mathcal{B}^i\phi_1)$ and $(\mathcal{B}^i\phi_1, \mathcal{B}^{i+1}\phi_0)$, respectively, for $0 \leq i \leq l$, where $\mathcal{B}$ is given by (3.16).

For any $(\varphi_0, \varphi_1) \in \Xi_0^{2(k+1)}(\Omega) \times \Xi_0^{2k+1}(\Omega)$, let φ be the solution of the problrm (3.15) with the initial data (φ_0, φ_1). Then $\varphi^{(2i)}$ and $\varphi^{(2i+1)}$ are the solutions of the problem (3.15) corresponding to the initial data $(\mathcal{B}^i\varphi_0, \mathcal{B}^i\varphi_1)$ and $(\mathcal{B}^i\varphi_1, \mathcal{B}^{i+1}\varphi_0)$, respectively, for $0 \leq i \leq k$. Using the initial data (ϕ_0, ϕ_1) and $(\mathcal{B}^{k+1}\varphi_0, \mathcal{B}^{k+1}\varphi_1)$ in the formula (3.27), we obtain

$$\Big(\psi(0), \mathcal{B}^{k+1}\varphi_1\Big) - \Big(\dot{\psi}(0),\ \mathcal{B}^{k+1}\varphi_0\Big) = -\int_{\wp_0} z(t)\phi_{\nu_A}\varphi_{\nu_A}^{(2k+2)} d\wp. \tag{3.30}$$

In one hand, by integration by parts with respect to the variable t on $[0, T]$, we obtain

$$\begin{aligned}
&-\int_{\wp_0} z(t)\phi_{\nu_A}\varphi_{\nu_A}^{(2k+2)} d\wp \\
&= \sum_{j=1}^{k} (-1)^j \phi_{\nu_A}^{(j)}(0)\varphi_{\nu_A}^{(2k+1-j)}(0) + \int_{\wp_0} \left(z(t)\phi_{\nu_A}\right)^{(k+1)} \varphi_{\nu_A}^{(k+1)} d\wp \\
&= \sum_{j=0}^{l} \Big(\mathcal{B}^j\phi_0\Big)_{\nu_A} \Big(\mathcal{B}^{k-j}\varphi_1\Big)_{\nu_A} - \sum_{j=0}^{l-1} \Big(\mathcal{B}^j\phi_1\Big)_{\nu_A} \Big(\mathcal{B}^{k-j}\varphi_0\Big)_{\nu_A} + I(\phi, \varphi),
\end{aligned} \tag{3.31}$$

where

$$I(\phi, \varphi) = \sum_{j=1}^{k+1} \int_{\wp_0} z^{(j)}(t)\phi_{\nu_A}^{(k+1-j)}\varphi_{\nu_A}^{(k+1)} d\wp + \int_{\wp_0} z(t)\phi_{\nu_A}^{(k+1)}\varphi_{\nu_A}^{(k+1)} d\wp. \tag{3.32}$$

On the other hand, using the formula (3.17), the boundary conditions (3.24), and the equation (3.26), we obtain

$$\begin{aligned}
\Big(\psi(0), \mathcal{B}^{k+1}\varphi_1\Big) &= \Big((\mathcal{B}^\star)^l\psi(0), \mathcal{B}^{l+1}\varphi_1\Big) + \sum_{j=0}^{l-1} \int_\Gamma (\mathcal{B}^\star)^j\psi(0) \Big(\mathcal{B}^{k-j}\varphi_1\Big)_{\nu_A} d\Gamma \\
&= -\Big(A\nabla(\mathcal{B}^\star)^l\psi(0), \nabla\mathcal{B}^l\varphi_1\Big) + \sum_{j=0}^{l} \int_{\Gamma_0} \psi^{(2j)}(0) \Big(\mathcal{B}^{k-j}\varphi_1\Big)_{\nu_A} d\Gamma \\
&\quad - \Big((\mathcal{B}^\star)^l\psi(0), F(\mathcal{B}^l\varphi_1) + (\mathcal{B}^l\varphi_1)\operatorname{div} F\Big),
\end{aligned} \tag{3.33}$$

and

$$\left(\dot\psi(0),\mathcal{B}^{k+1}\varphi_0\right) \;=\left((\mathcal{B}^\star)^l\dot\psi(0),\mathcal{B}^{l+1}\varphi_0\right)+\sum_{j=0}^{l-1}\int_{\Gamma_0}\psi^{(2j+1)}(0)\left(\mathcal{B}^{k-j}\varphi_0\right)_{\nu_A}d\Gamma. \quad (3.34)$$

Noting that $\psi^{(2j)}(0)=\phi^{(2j)}(0)=\mathcal{B}^j\phi_0$ and $\psi^{(2j+1)}(0)=\mathcal{B}^j\phi_1$ on Γ_0 and using (3.30)-(3.34), we have the following identity

$$\begin{aligned}&-\left(A\nabla(\mathcal{B}^\star)^l\psi(0),\nabla\mathcal{B}^l\varphi_1\right)-\left((\mathcal{B}^\star)^l\dot\psi(0),\mathcal{B}^{l+1}\varphi_0\right)\\&=I(\phi,\varphi)+\left((\mathcal{B}^\star)^l\psi(0),\;F(\mathcal{B}^l\varphi_1)+(\mathcal{B}^l\varphi_1)\operatorname{div}F\right)\end{aligned}\quad (3.35)$$

Since $\Xi_0^{2(k+1)}(\Omega)\times\Xi_0^{2k+1}(\Omega)$ is dense in $\Xi_0^{k+2}(\Omega)\times\Xi_0^{k+1}(\Omega)$, the identity (3.35) is actually true for all $(\varphi_0,\varphi_1)\in\Xi_0^{k+2}(\Omega)\times\Xi_0^{k+1}(\Omega)$.

Letting $\varphi_0=0$ in (3.35), we obtain

$$\left(\mathcal{A}(\mathcal{B}^\star)^l\psi(0),\mathcal{B}^l\varphi_1\right)=I(\phi,\varphi)+\left((\mathcal{B}^\star)^l\psi(0),\;F(\mathcal{B}^l\varphi_1)+(\mathcal{B}^l\varphi_1)\operatorname{div}F\right), \quad (3.36)$$

for $\varphi_1\in\Xi_0^{k+1}(\Omega)$ where φ is the solution of the problem (3.15) for the initial data $(0,\varphi_1)$. It is easy to check by the maximum principle for the elliptic operator that

$$\overline{Image(\mathcal{B}^l)}=L^2(\Omega). \quad (3.37)$$

Moreover, by virtue of the inequality (3.22) and Lemma 3.1, we have the estimate

$$\begin{aligned}&|I(\phi,\varphi)|\\&\le c\sum_{j=0}^{l}\left[\int_0^T\int_{\Gamma_0}\left((\phi_{\nu_A}^{(2j)})^2+(\phi_{\nu_A}^{(2j+1)})^2\right)d\Gamma dt\right]^{1/2}\left(\int_{\wp_0}(\varphi_{\nu_A}^{(k+1)})^2d\wp\right)^{1/2}\\&\le c\sum_{j=0}^{l}\left(E(\mathcal{B}^j\phi_0,\mathcal{B}^j\phi_1)+E(\mathcal{B}^j\phi_1,\mathcal{B}^{j+1}\phi_0)\right)^{1/2}\|A^{1/2}\nabla\mathcal{B}^l\varphi_1\|\\&\le c\left(\|\phi_0\|_{k+2}^2+\|\phi_1\|_{k+1}^2\right)^{1/2}\|\mathcal{B}^l\varphi_1\|_1.\end{aligned}\quad (3.38)$$

In terms of (3.36)-(3.38), we obtain

$$\begin{aligned}\|\mathcal{A}(\mathcal{B}^\star)^l\psi(0)\|_{-1}\;&=\sup_{\|\mathcal{B}\varphi_1\|_1=1}\left(\mathcal{A}(\mathcal{B}^\star)^l\psi(0),\;\mathcal{B}^l\varphi_1\right)\\&\le c\left(\|\phi_0\|_{k+2}^2+\|\phi_1\|_{k+1}^2\right)^{1/2}+c\|\psi(0)\|_k.\end{aligned}\quad (3.39)$$

Furthermore, on the boundary Γ the problem (3.26) implies

$$\begin{aligned}\|(\mathcal{B}^\star)^i\psi(0)\|_{H^{k+1/2-2i}(\Gamma)}\;&=\|\psi^{(2i)}(0)\|_{H^{k+1/2-2i}(\Gamma)}=\|\phi_{\nu_A}^{(2i)}(0)\|_{H^{k+1/2-2i}(\Gamma_0)}\\&=\|(\mathcal{B}^i\phi_0)_{\nu_A}\|_{H^{k+1/2-2i}(\Gamma_0)}\le c\|\phi_0\|_{k+2},\quad 0\le i\le l.\end{aligned}\quad (3.40)$$

Now, using the ellipticity of the operator $\mathcal{B}^\star$ and from (3.39) and (3.40), we have

$$\begin{aligned}\|\psi(0)\|_{k+1} &\le c\|\mathcal{B}^\star\psi\|_{k-1} + c\|\psi(0)\|_{H^{k+1/2}(\Gamma)} + c\|\psi(0)\|_k \\ &\le c\|(\mathcal{B}^\star)^2\psi\|_{k-3} + c\|\mathcal{B}^\star\psi(0)\|_{H^{k+1/2-2}(\Gamma_0)} + \|\phi_0\|_{k+2} + c\|\psi(0)\|_k \\ &\le c\|(\mathcal{B}^\star)^l\psi(0)\|_{-1} + c\|\phi_0\|_{k+2} + c\|\psi(0)\|_k \\ &\le c\left(\|\phi_0\|_{k+2}^2 + \|\phi_1\|_{k+1}^2\right)^{1/2},\end{aligned} \tag{3.41}$$

where the induction assumption $\|\psi(0)\|_k \le c\left(\|\phi_0\|_{k+1}^2 + \|\phi_1\|_k^2\right)^{1/2}$ is used.

A similar argument yields

$$\|\dot\psi(0)\|_k \le c\left(\|\phi_0\|_{k+2}^2 + \|\phi_1\|_{k+1}^2\right)^{1/2}, \tag{3.42}$$

after we let $\varphi_0 \in \Xi_0^{k+2}(\Omega)$ and $\varphi = 0$ in (3.35).

Next, let us prove the left hand side of the inequality (3.28) where k is replaced by $k+1$. We set $\varphi_0 = \phi_0$ and $\varphi_1 = \phi_1$ in (3.35) and use Lemma 3.1 to obtain

$$\begin{aligned}& c\left(\|\psi(0)\|_{k+1}^2 + \|\dot\psi(0)\|_k^2\right)^{1/2} E^{1/2}(\mathcal{B}^l\phi_1, \mathcal{B}^{l+1}\phi_0) \\ &\ge I(\phi,\phi) - c_1\|\psi(0)\|_k\|A^{1/2}\nabla\mathcal{B}^l\phi_1\| \\ &\ge \int_0^{T_1}\int_{\Gamma_0}\left(\phi_{\nu_A}^{(k+1)}\right)^2 d\Gamma dt - \varepsilon\int_{\wp_0}\left(\phi_{\nu_A}^{(k+1)}\right)^2 d\wp - c_\varepsilon\sum_{j=0}^{k}\int_{\wp_0}\left(\phi_{\nu_A}^{(j)}\right)^2 d\wp \\ &\quad -\varepsilon\|\psi(0)\|_k^2 - c_\varepsilon\|A^{1/2}\nabla\mathcal{B}^l\phi_1\|^2 \\ &\ge c_1 E(\mathcal{B}^l\phi_1, \mathcal{B}^{l+1}\phi_0) - c_2\left(\|\phi_0\|_{k+1}^2 + \|\phi_1\|_k^2\right).\end{aligned} \tag{3.43}$$

In addition, $(\phi_0, \phi_1) \in \Xi_0^{k+2}(\Omega) \times \Xi_0^{k+1}(\Omega)$ implies, by the ellipticity of the operator $\mathcal{B}$,

$$\|\phi_0\|_{k+2}^2 + \|\phi_1\|_{k+1}^2 \le cE(\mathcal{B}^l\phi_1, \mathcal{B}^{l+1}\phi) + c\left(\|\phi_0\|_{k+1}^2 + \|\phi_1\|_k^2\right). \tag{3.44}$$

Then the inequalities (3.43) and (3.44) give, via the induction assumption $\|\psi(0)\|_k^2 + \|\dot\psi(0)\|_{k-1}^2 \ge c\left(\|\phi_0\|_{k+1}^2 + \|\phi_1\|_k^2\right)$,

$$\|\psi(0)\|_{k+1}^2 + \|\dot\psi(0)\|_k^2 \ge c\left(\|\phi_0\|_{k+2}^2 + \|\phi_1\|_{k+1}^2\right). \tag{3.45}$$

The relations (3.37), (3.42) and (3.45) mean that the inequality (3.28) is true with k replaced by $k+1$ if $k = 2l$ for some $l \ge 1$.

Case II If $k = 2l+1$, a similar argument can establish the inequality (3.28) where k is replaced by $k = 1$.

Then Lemma 3.2 follows by induction.

Lemma 3.3 *Let ϕ solve the problem* (3.15) *with the initial data* $(\phi_0, \phi_1) \in \Xi_0^2(\Omega) \times \Xi_0^1(\Omega)$. *Then*

$$\phi_{\nu_A} \in C\left([0,T], H^{1/2}(\Gamma)\right) \cap H^1\left((0,T)\times\Gamma\right). \tag{3.46}$$

Proof. For any $T > 0$ given, there is $c_T > 0$ such that

$$\|\phi(t)\|_2 \le c_T \left(\|\phi_0\|_2^2 + \|\phi_1\|_1^2\right) \quad \forall\, t \in [0, T],$$

which implies $\phi_{\nu_A} \in C\left([0, T], H^{1/2}(\Gamma)\right)$.

Since ϕ' is the solution of the problem (3.15) for the initial data $(\phi_1, \mathcal{B}\phi_0) \in H_0^1(\Omega) \times L^2(\Omega)$, Lemma 3.1 implies $\phi'_{\nu_A} \in L^2((0, T) \times \Gamma)$.

To complete the proof, it is remaining to show that $\phi_{\nu_A} \in L^2((0, T), H^1(\Gamma))$.

Let X be a vector field of the manifold Γ, that is, $X(x) \in \Gamma_x$ for each $x \in \Gamma$. We extend X to the whole $\overline{\Omega}$ to be a vector field on the manifold $(\overline{\Omega}, g)$ where $g = A^{-1} = (a_{ij}(x, \nabla w))^{-1}$.

Let

$$v = X(\phi), \quad (t, x) \in (0, T) \times \Omega. \tag{3.47}$$

Then v solves the problem

$$\begin{cases} \ddot{v} = \mathcal{B}v + [X, \mathcal{B}]\phi, \quad (0, T) \times \Omega, \\ v|_\Gamma = 0, \quad t \in (0, T), \\ v(0) = X(\phi_0) \in H_0^1(\Omega), \quad \dot{v}(0) = X(\phi_1) \in L^2(\Omega), \end{cases} \tag{3.48}$$

where $[X, \mathcal{B}]\phi = X(\mathcal{B}\phi) - \mathcal{B}X(\phi)$ with the estimate

$$\|[X, \mathcal{B}]\phi(t)\| \le c_T \left(\|\phi_0\|_2^2 + \|\phi_1\|_1^2\right) \quad \forall\, t \in [0, T]. \tag{3.49}$$

Let H be a vector field on $\overline{\Omega}$ with

$$H(x) = \nu_A \quad x \in \Gamma.$$

We multiply the both sides of the equation in (3.48) by $H(v)$ and integrate by parts over $\Sigma = (0, T) \times \Omega$ to obtain

$$\begin{aligned} &\int_\wp \left[v_{\nu_A}^2 - \frac{1}{2}|\nabla_g v|_g^2 |\nu_A|_g^2\right] d\wp \\ &= (\dot{v}, H(v))\,|_0^T + \int_\Sigma \left[D_g H(\nabla_g v, \nabla_g v) + \frac{1}{2}\left(\dot{v}^2 - |\nabla_g v|_g^2\right) \operatorname{div} H\right] d\Sigma \\ &\quad + \left(F(v) + v \operatorname{div} F - [X, \mathcal{B}]\phi, \;\; H(v)\right), \end{aligned} \tag{3.50}$$

where $\wp = (0, T) \times \Gamma$. In addition, the boundary condition $v|_\Gamma = 0$ implies

$$|\nabla_g v|_g^2 = \frac{1}{|\nu_A|_g^2} v_{\nu_A}^2 \quad \forall\, x \in \Gamma. \tag{3.51}$$

In terms of (3.50), (3.51) and (3.49), we obtain

$$\int_\wp v_{\nu_A}^2 d\wp \le c_T \left(\|\phi_0\|_2^2 + \|\phi_1\|_1^2\right). \tag{3.52}$$

Since

$$v_{\nu_A} = \nu_A\left(X(\phi)\right) = X(\phi_{\nu_A}) + [\nu_A, X]\phi \quad x \in \Gamma,$$

by (3.52), we have

$$\int_0^T \int_\Gamma |X(\phi_{\nu_A})|^2 d\Gamma dt \le c_{T,X}\left(\|\phi_0\|_2^2 + \|\phi_1\|_1^2\right),$$

for any vector field X of the manifold Γ, that is, $\phi_{\nu_A} \in L^2\left((0,T), H^1(\Gamma)\right)$. $\|$

The Proof of Theorem 3.1 Let $(v_0, v_1) \in \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^{m-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right)$ be given. By Lemma 3.2, there is $(\phi_0, \phi_1) \in \Xi_0^{m+1}(\Omega) \times \Xi_0^m(\Omega)$ such that the control $\varphi = z\phi_{\nu_A}$ on $\wp_0 = (0,T) \times \Gamma_0$ drives the system (3.12) to rest at the time T, where ϕ is the solution of the problem (3.15) with the initial data (ϕ_0, ϕ_1).

Since $\phi^{(k)}$ are the solutions of the problem (3.15) with the initial data

$$\begin{cases} (\mathcal{B}^l\phi_0, \mathcal{B}^l\phi_1) & \text{if} \quad k = 2l \text{ or,} \\ (\mathcal{B}^l\phi_1, \mathcal{B}^{l+1}\phi_0) & \text{if} \quad k = 2l+1, \end{cases} \tag{3.53}$$

for $0 \le k \le m-1$, Lemma 3.3 implies $\varphi = z\phi_{\nu_A} \in \tilde{\mathcal{X}}_0^m(\Omega)$.

4 Locally exact controllability; the Neumann action

Let $T > 0$ be given. This time, we introduce a Banach space $\mathcal{X}^m_{0N}(T)$ as follows. $\mathcal{X}^m_{0N}(T)$ consists of all the functions

$$\varphi \in \cap_{k=0}^{m-2} C^k\left([0,T], H^{m-k-3/2}(\Gamma_0)\right), \quad \varphi^{(k)} \in L^2\left((0,T), H^{1/2}(\Gamma_0)\right), \tag{4.1}$$

$$\varphi^{(k)}(0) = 0, \quad x \in \Gamma_0, \quad 0 \le k \le m-1, \tag{4.2}$$

with the norm

$$\|\varphi\|^2_{\mathcal{X}^m_{0N}(T)} = \sum_{k=0}^{m-2} \|\varphi^{(k)}\|^2_{C\left([0,T], H^{m-1/2-k}(\Gamma_0)\right)} + \sum_{k=0}^{m-1} \|\varphi^{(k)}\|^2_{L^2\left((0,T), H^{1/2}(\Gamma_0)\right)}. \tag{4.3}$$

Let $w \in H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be given. We invoke Theorem 1.4 to define a map for $\varphi \in \mathcal{X}^m_{0N}(T)$ by setting

$$\Phi_N(\varphi) = (u(T), \dot{u}(T)), \tag{4.4}$$

where u is the solution of the following problem

$$\begin{cases} \ddot{u} = \operatorname{div} \mathbf{a}(x, \nabla u) \quad (t,x) \in (0,T) \times \Omega, \\ u|_{\Gamma_1} = 0, \quad t \in (0,T), \\ \langle \mathbf{a}(x, \nabla u), \nu\rangle|_{\Gamma_0} = \langle \mathbf{a}(x, \nabla w), \nu\rangle + \varphi, \quad t \in (0,T), \\ u(0) = w, \quad \dot{u}(0) = 0. \end{cases} \tag{4.5}$$

Let $\varepsilon_T > 0$ be given by Theorem 1.4. Then

$$\Phi_N : \quad B_{\mathcal{X}_{0N}^m(T)}(0, \varepsilon_T) \to \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^{m-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right), \tag{4.6}$$

where $B_{\mathcal{X}_{0N}^m(T)}(0, \varepsilon_T) \subset \mathcal{X}_{0N}^m(T)$ is the ball with the radius ε_T centered at 0.

We observe that, since $w \in H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$,

$$\Phi_N(0) = (w, 0) \in \left(H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right).$$

Then Theorem 1.5 is equivalent to the following claim: For some $T > 0$ there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ with $\varepsilon_T \geq \varepsilon_2$ such that

$$B_{H^{m+1}(\Omega) \times H^m(\Omega)}\left((w, 0), \varepsilon_1\right) \subset \Phi_N\left(B_{\mathcal{X}_{0N}^m(T)}(0, \varepsilon_2)\right), \tag{4.7}$$

where $B_{H^{m+1}(\Omega) \times H^m(\Omega)}\left((w, 0), \varepsilon_1\right)$ is the ball with the radius ε_1 centered at 0 in the space $H^{m+1}(\Omega) \times H^m(\Omega)$.

The map Φ_N is *Fréchet* differentiable on $B_{\mathcal{X}_{0N}^m(T)}(0, \varepsilon_T)$. In particular,

$$\Phi_N'(0)\varphi = (v(T), \dot{v}(T)), \quad \varphi \in \mathcal{X}_{0N}^m(T), \tag{4.8}$$

where $v(t, x)$ is the solution of the linear system with variable coefficients in the space variable

$$\begin{cases} \ddot{v} = \operatorname{div} A(x, \nabla w)\nabla v, \quad (t, x) \in (0, T) \times \Omega, \\ v|_{\Gamma_1} = 0, \quad t \in (0, T), \\ v_{\nu_A}|_{\Gamma_0} = \varphi, \quad t \in (0, T), \\ v(0) = \dot{v}(0) = 0, \end{cases} \tag{4.9}$$

where $v_{\nu_A} = \langle A(x, \nabla w)\nabla v, \, \nu\rangle$.

The proof of the exact controllability with the Neumann action depends on the fact:

Proposition 4.1 *Let $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$, and $\mathcal{Y}_2$ be Banach spaces with $\mathcal{X}_2 \subset \mathcal{X}_1$, $\mathcal{Y}_2 \subset \mathcal{Y}_1$, $\overline{\mathcal{X}_2} = \mathcal{X}_1$, and $\overline{\mathcal{Y}_2} = \mathcal{Y}_1$. Suppose that $\Phi : \mathrm{B}_{\mathcal{X}_i}(0, r) \to \mathcal{Y}_i$ are mappings of C^1 for $i = 1$, 2 such that*

$$\mathcal{Y}_2 \subset \Phi'(0)\mathcal{X}_1. \tag{4.10}$$

There is $\varepsilon > 0$ such that

$$\mathrm{B}_{\mathcal{Y}_2}(\Phi(0), \varepsilon) \subset \Phi\left(\mathrm{B}_{\mathcal{X}_1}(0, r)\right). \tag{4.11}$$

Proof. Let $y_0 = \Phi(0)$. It will suffice to prove that for any y in $\mathcal{Y}_2$ near y_0, the equation

$$\Phi(x) - \Phi(0) = y - y_0 \tag{4.12}$$

has a solution x in $\mathrm{B}_{\mathcal{X}_1}(0, r)$. This can be done by a modification of the proof of Theorem (3.1.19) in Berger [1].

We denote by $\mathcal{X}_1/\ker\Phi'(0)$ the quotient space where

$$\ker\Phi'(0) = \{\, x \,|\, x \in \mathcal{X}_1, \Phi'(0)x = 0\,\}.$$

The assumptions (4.10) imply that the inversion of $\Phi'(0)$: $\mathcal{Y}_2 \to \mathcal{X}_1/\ker\Phi'(0)$ exists, is closed, and therefore is bounded. Then, there is $C > 0$ such that

$$C\|\Phi'(0)x\|_{\mathcal{Y}_2} \geq d\,(x, \ker\Phi'(0)) \tag{4.13}$$

for $x \in \mathcal{X}_1$ such that $\Phi'(0)x \in \mathcal{Y}_2$ where $d\,(x, \ker\Phi'(0))$ is the distance from x to the space $\ker\Phi'(0)$ in $\mathcal{X}_1$.

Now we can construct a sequence $\{\, x_k \,\}$ as follows. Let $\varepsilon > 0$ be given. Let

$$R(x) = \Phi(x) - \Phi(0) - \Phi'(0)x, \quad x \in \mathrm{B}_{\mathcal{X}}(0, r). \tag{4.14}$$

Since Φ: $\mathrm{B}_{\mathcal{X}_2}(0,r) \to \mathcal{Y}_2$ is C^1 and $\overline{\mathcal{X}_2} = \mathcal{X}_1$, we take $x_0 \in \mathrm{B}_{\mathcal{X}_1}(0,r) \cap \mathcal{X}_2$. Then $R(x_0) \in \mathcal{Y}_2$. Next, the relations (4.10) and (4.13) imply that there is $x{*}_1 \in \mathcal{X}_1$ such that

$$\Phi'(0)x{*}_1 = y - y_0 - R(x_0), \tag{4.15}$$

$$\|x{*}_1\|_{\mathcal{X}_1} \leq C\|y - y_0 - R(x_0)\|_{\mathcal{Y}_2}. \tag{4.16}$$

If $x{*}_1 = x_0$, then x_0 is a solution to the equation (4.12) and the constructing ends. We assume that $x{*}_1 \neq x_0$. We take $x_1 \in \mathcal{X}_2$ such that

$$\|x_1 - x{*}_1\|_{\mathcal{X}_1} \leq \varepsilon\|x_1 - x_0\|_{\mathcal{X}_1}, \tag{4.17}$$

$$\|x_1\|_{\mathcal{X}_1} \leq C\|y - y_0 - R(x_0)\|_{\mathcal{Y}_2}. \tag{4.18}$$

Proceeding this procedure, we obtain two sequences $\{\, x{*}_k \,\} \subset \mathcal{X}_1$ and $\{\, x_k \,\} \subset \mathcal{X}_2$ satisfying

$$\Phi'(0)x{*}_k = y - y_0 - R(x_{k-1}), \tag{4.19}$$

$$\|x_k\|_{\mathcal{X}_1} \leq C\|y - y_0 - R(x_{k-1})\|_{\mathcal{Y}_2}, \tag{4.20}$$

$$\|x_k - x{*}_k\|_{\mathcal{X}_1} \leq \varepsilon\|x_k - x_{k-1}\|_{\mathcal{X}_1}, \tag{4.21}$$

for $k \geq 1$.

A similar argument as in the proof of Theorem (3.1.19) in Berger [1] completes the proof. ‖

For $i = 1$, 2, let

$$\mathcal{X}_i = \mathcal{X}_{0N}^{m+i-1}(T),$$
$$\mathcal{Y}_i = \left(H^{m+i-1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^{m+i-2}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right).$$

It is easy to check by Theorem 2.2 that the mappings Φ_N, given by (4.4), are of C^1 from $\mathrm{B}_{\mathcal{X}_i}(0,r)$ to $\mathcal{Y}_i$ for $i = 1, 2$, and for some $r > 0$. By Proposition 4.1, to prove Theorem 1.5 is to establish the exact controllability of the system (4.9) on the space $\left(H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right)$, which for a reversible system such as (4.9) is equivalent to null controllability.

Explicitly, one has to show that, for specified T, given $(v_0, v_1) \in \left(H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right)$, one can find $\varphi \in \tilde{\mathcal{X}}^m_{0N}(T)$ such that the solution to

$$\begin{cases} \ddot{v} = \operatorname{div} A(x, \nabla w)\nabla v, \quad (t,x) \in (0,T) \times \Omega, \\ v|_{\Gamma_1} = 0, \quad t \in (0,T), \\ v_{\nu_A}|_{\Gamma_0} = \varphi, \quad t \in (0,T), \\ v(0) = v_0, \quad \dot{v}(0) = v_1, \end{cases} \tag{4.22}$$

satisfies

$$v(T) = \dot{v}(T) = 0, \tag{4.23}$$

where $\tilde{\mathcal{X}}^m_0(T)$ is the Banach space of all function with (4.1) and the norm (4.3) but with (4.2) replaced by

$$\varphi^{(k)}(T) = 0, \quad x \in \Gamma_0, \quad 0 \le k \le m-1. \tag{4.24}$$

Then Theorem 1.5 follows by the following

Theorem 4.1 *Let an equilibrium $w \in H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable. Then there exists a $T_0 > 0$ such that for any $T > T_0$ and*

$$(v_0, v_1) \in \left(H^{m+1}(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right) \times \left(H^m(\Omega) \cap H^1_{\Gamma_1}(\Omega)\right),$$

there is a $\varphi \in \tilde{\mathcal{X}}^m_{0N}(T)$ such that the solution

$$v \in \cap_{k=0}^m C^k\left([0,T], H^{m-k}(\Omega)\right)$$

of the problem (4.22) *satisfies* (4.23).

As in Section 3, we shall work out the smooth control from the distributed control theory.

We start with the dual system of the problem (4.22)

$$\begin{cases} \ddot{\phi} = \operatorname{div} A(x, \nabla w)\nabla \phi, \quad (t,x) \in (0,T) \times \Omega, \\ \phi|_{\Gamma_1} = \phi_{\nu_A}|_{\Gamma_0} = 0, \quad t \in (0,T), \\ \phi(0) = \phi_0, \quad \dot{\phi}(0) = \phi_1. \end{cases} \tag{4.25}$$

We shall need the following observability inequality to get rid of a lower order term in Lemma 4.4 later: There exists a $T_1 > 0$ such that for any $T > T_1$, there is a constant $c_T > 0$ for which

$$c_T \int_{\wp_0} \dot{\phi}^2 d\wp \ge E(\phi_0, \phi_1), \tag{4.26}$$

where ϕ is the solution of the problem (4.25) and

$$\wp_0 = (0,T)\times\Gamma_0, \quad E(\phi_0,\phi_1) = \|A^{1/2}\nabla\phi_0\|^2 + \|\phi_1\|^2, \quad (\phi_0,\phi_1)\in H^1_{\Gamma_1}(\Omega)\times L^2(\Omega),$$

whenever the left-hand side is finite.

The inequality (4.26) was established by Lasiecka and Triggiani [13] for the classical Laplacian where $\operatorname{div} A(x,\nabla w)\nabla\phi = \Delta\phi$ and was extended to the case of the variable coefficients with a first order term in Lasiecka, Triggiani, and Yao [15], under some geometrical conditions.

The lemma below follows by Lasiecka, Triggiani, and Yao [15], Theorem 3.2, where a uniqueness result, needed, is given by Triggiani and Yao [23], Theorem 10.1.1.

Lemma 4.1 *Let $w\in H^{m+1}(\Omega)\cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable such that the assumption* (1.10) *is true and let Γ_1 be such that* (1.14) *holds. There exists a $T_1>0$ such that for any $T>T_1$, there is a constant $c_T>0$ for which the inequality* (4.26) *is true whenever the left-hand side is finite.*

However, to find out the smooth control, one-side observability estimates, as in (4.26), are insufficient. We have to seek to establish boundary estimates of another type controlled by the initial energy both sides from below and also from above, as in (3.22).

Let $\varepsilon>0$ be given small. Let $\eta_\varepsilon\in C^\infty(\mathcal{R})$ be such that $0\le\eta_\varepsilon\le 1$ and

$$\eta_\varepsilon(t)=1 \quad t\le-\varepsilon; \quad \eta_\varepsilon(t)=0 \quad t\ge 0.$$

For any $T>\varepsilon$, let

$$z(t)=\eta_\varepsilon(t-T). \tag{4.27}$$

Then

$$z(t)=1 \quad 0\le t\le T-\varepsilon; \quad z(t)=0 \quad t\ge T.$$

Lemma 4.2 *Let*

$$g=A^{-1}(x,\nabla w) \tag{4.28}$$

be the Riemannian metric on $\overline{\Omega}$. Let ϕ solve the problem

$$\ddot\phi = \operatorname{div} A(x,\nabla w)\nabla\phi \quad (t,x)\in\Sigma, \tag{4.29}$$

where $\Sigma=(0,T)\times\Omega$. Let H be a vector field on $\overline{\Omega}$ and $P\in C^2(\overline{\Omega})$ be a function. Then

$$\begin{aligned}&\int_\wp z\left[H(\phi)\phi_{\nu_A}+\frac12(\dot\phi^2-|\nabla_g\phi|_g^2)\langle H,\nu\rangle\right]d\wp\\ &=-(\phi_1,H(\phi_0))-\int_{T-\varepsilon}^T \dot z\left(\dot\phi,H(\phi)\right)dt\\ &\quad+\int_\Sigma z\left[D_gH(\nabla_g\phi,\nabla_g\phi)+\frac12(\dot\phi^2-|\nabla_g\phi|^2)\operatorname{div}H\right]d\Sigma,\end{aligned} \tag{4.30}$$

where $\wp = (0,T)\times\Gamma$, *and*

$$\begin{aligned}&\int_\Sigma zP\left(\dot\phi^2 - |\nabla_g\phi|_g^2\right)d\Sigma\\&= -(\phi_1, P\phi_0) - \int_{T-\varepsilon}^T \dot z(\dot\phi, P\phi)dt - \frac{1}{2}\int_\Sigma z\phi^2\mathcal{A}P d\Sigma\\&\quad + \int_\wp z\left[\frac{1}{2}\phi^2 P_{\nu_A} - P\phi\phi_{\nu_A}\right]d\wp.\end{aligned}\tag{4.31}$$

Proof. We multiply the equation (4.29) by $zH(\phi)$ and $zP\phi$, respectively, integrate by parts over $\Sigma = (0,T)\times\Omega$, and obtain the identities (4.30) and (4.31), see Yao [24], Proposition 2.1. $\|$

Let

$$\Psi(\varphi,\phi) = \int_{\wp_0} z\left(\dot\varphi\dot\phi - \langle\nabla_{\Gamma_g}\varphi, \nabla_{\Gamma_g}\phi\rangle_g\right)h_0 d\wp,\tag{4.32}$$

where φ and ϕ solve the problem (4.25) with the initial data (φ_0,φ_1) and (ϕ_0,ϕ_1), respectively, and

$$h_0 = \langle H_0, \nu\rangle,\quad H_0 = 2\rho_g\nabla_g\rho_g,\quad x\in\Gamma,\tag{4.33}$$

and ρ_g is the distance function of the Riemannian metric g in (4.28).

The second observability estimate we need is the following

Lemma 4.3 *Let* $w\in H^{m+1}(\Omega)\cap H^1_{\Gamma_1}(\Omega)$ *be exactly controllable such that the assumption* (1.10) *is true and let* Γ_1 *be such that* (1.14) *holds. Let* $\varepsilon>0$ *be given small. There are constant* $c_{\varepsilon1}>0$, $c_{\varepsilon2}>0$, *and* $c_0>0$, *independent of time* t *and solutions* ϕ *of the problem* (4.25), *such that for any* $T>\varepsilon$

$$\begin{gathered}c_{\varepsilon2}TE(\phi_0,\phi_1) \ge \Psi(\phi,\phi) + c_0\int_{\wp_0}\phi^2 d\wp + c_0\int_\Sigma \phi^2 d\Sigma \ge \left[\rho_0(T-\varepsilon) - c_{\varepsilon1}\right]E(\phi_0,\phi_1),\\ \textit{for all}\quad (\phi_0,\phi_1)\in H^1_{\Gamma_1}(\Omega)\times L^2(\Omega),\end{gathered}\tag{4.34}$$

where $\rho_0>0$ *is given in* (1.10).

Proof. We take $P = \operatorname{div} H_0 - \rho_0$ in the identity (4.31) and obtain the estimate

$$\begin{aligned}&\int_\Sigma z(\operatorname{div} H_0 - \rho_0)\left(\dot\phi^2 - |\nabla_g\phi|_g^2\right)d\Sigma\\&\ge -c_{\varepsilon1}E(\phi_0,\phi_1) - c_0\int_\Sigma \phi^2\Sigma - c_0\int_\wp z\phi^2\wp,\end{aligned}\tag{4.35}$$

where the boundary conditions $\phi|_{\Gamma_1} = \phi_{\nu_A}|_{\Gamma_0} = 0$ are used.

Let us take $H = H_0$ in the identity (4.30) to check the boundary terms on the left-hand side of the identity (4.30). On Γ_1, $\phi_{\Gamma_1} = 0$ implies

$$H_0(\phi) = \frac{h_0}{|\nu_A|_g^2}\phi_{\nu_A},\quad |\nabla_g\phi|_g^2 = \frac{1}{|\nu_A|_g^2}\phi^2_{\nu_A},$$

which implies with $h_0 \le 0$ for $x \in \Gamma_1$ together that

$$\int_{\wp_1} z \left[H_0(\phi)\phi_{\nu_A} + \frac{1}{2}(\dot{\phi}^2 - |\nabla_g \phi|_g^2) h_0 \right] d\wp = \frac{1}{2} \int_{\wp_1} z \frac{\phi_{\nu_A}^2}{|\nu_A|_g^2} h_0 d\wp \le 0; \tag{4.36}$$

On $\wp_0 = (0, T) \times \Gamma_0$, $\phi_{\nu_A} = 0$ implies $\nabla_g \phi = \nabla_{\Gamma_g} \phi$. We then have via the identity (4.30) where $H = H_0$ and (4.35)-(4.36), (1.10), that

$$\begin{aligned} \Psi(\phi, \phi) &\ge 2\rho_0 \int_\Sigma z |\nabla_g \phi|_g^2 d\Sigma - c_{\varepsilon 1} E(\phi_0, \phi_1) \\ &\quad + \int_\Sigma z(\dot{\phi}^2 - |\nabla_g \phi|^2) \operatorname{div} H_0 d\Sigma \\ &\ge \rho_0 \int_\Sigma z(\dot{\phi}^2 + |\nabla_g \phi|_g^2) d\Sigma - c_{\varepsilon 1} E(\phi_0, \phi_1) \\ &\quad + \int_\Sigma z(\dot{\phi}^2 - |\nabla_g \phi|^2)(\operatorname{div} H_0 - \rho_0) d\Sigma \\ &\ge [\rho_0(T - \varepsilon) - c_{\varepsilon 1}] E(\phi_0, \phi_1) \\ &\quad - c_0 \int_\Sigma \phi^2 d\Sigma - c_0 \int_{\wp_0} \phi^2 d\wp_0. \end{aligned} \tag{4.37}$$

On the other hand, since $\overline{\Gamma}_1 \cap \overline{\Gamma}_0 = \emptyset$, we take two open sets $\aleph_0$ and $\aleph_1$ in $\mathcal{R}^n$ such that $\aleph_0 \cap \aleph_1 = \emptyset$ and $\Gamma_i \subset \aleph_i$ for $i = 0$, 1, respectively. Let $h \in C^\infty(\mathcal{R}^n)$ be such that

$$h(x) = 1 \quad x \in \aleph_0; \quad h(x) = 0 \quad x \in \aleph_1.$$

Letting $H = hH_0$ in (4.30) yields

$$\Psi(\phi, \phi) \le c_{\varepsilon 2} T E(\phi_0, \phi_1). \tag{4.38}$$

The lemma follows by the inequalities (4.37) and (4.38). $\|$

We introduce an operator by

$$\mathcal{A}_0 v = \operatorname{div} A(x, \nabla w) \nabla v, \quad D(\mathcal{A}_0) = \{ v \in H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega), \, v_{\nu_A}|_{\Gamma_0} = 0 \}. \tag{4.39}$$

Let $(\phi_0, \phi_1) \in D(\mathcal{A}_0) \times H^1_{\Gamma_1}(\Omega)$. Then $(\phi_1, \mathcal{A}_0 \phi_0) \in H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$. Since $\dot{\phi}$ solves the problem (4.25) with the initial data $(\phi_1, \mathcal{A}_0 \phi_0)$, the inequality (4.34) implies

$$\begin{aligned} c_{\varepsilon 2} T E(\phi_1, \mathcal{A}_0 \phi_0) &\ge \Psi(\dot{\phi}, \dot{\phi}) + c_0 \int_{\wp_0} \dot{\phi}^2 d\wp + c_0 \int_\Sigma \dot{\phi}^2 d\Sigma \\ &\ge [\rho_0(T - \varepsilon) - c_{\varepsilon 1}] \, E(\phi_1, \mathcal{A}_0 \phi_1). \end{aligned} \tag{4.40}$$

Let T_1 and be given by Lemma 4.1. We fix $T_2 > T_1$. Let c_{T_2} be given by Lemma 4.1. It follows from Lemma 4.1 that for any $T > T_2 + \varepsilon$

$$\int_\Sigma \dot{\phi}^2 d\Sigma \le T E(\phi_0, \phi_1) \le T c_{T_2} \int_0^{T_2} \int_{\Gamma_0} \dot{\phi}^2 d\wp \le T c_{T_2} \int_{\wp_0} z \dot{\phi}^2 d\wp. \tag{4.41}$$

We introduce a bilinear form by

$$\Psi_*(\varphi,\phi)=\Psi(\varphi,\phi)+c_T\int_{\wp_0} z\varphi\phi d\wp, \tag{4.42}$$

where

$$c_T=c_0(1+Tc_{T_2}). \tag{4.43}$$

Then the inequalities (4.40) and (4.41) yield

Lemma 4.4 *For any $T>T_2+\varepsilon$ and $(\phi_0,\phi_1)\in D(\mathcal{A}_0)\times H^1_{\Gamma_1}(\Omega)$,*

$$c_{\varepsilon 2}TE(\phi_1,\mathcal{A}_0\phi_0)\geq\Psi_*(\dot\phi,\dot\phi)\geq[\rho_0(T-\varepsilon)-c_{\varepsilon 1}]\,E(\phi_1,\mathcal{A}_0\phi_0). \tag{4.44}$$

We now go back to the control problem in Theorem 4.1.

Given $(\phi_0,\phi_1)\in D(\mathcal{A}_0)\times H^1_{\Gamma_1}(\Omega)$, the problem (4.25) admits a unique solution. We then solve the problem

$$\begin{cases}\ddot\psi=\operatorname{div}A(x,\nabla w)\nabla\psi, & (t,x)\in(0,T)\times\Omega,\\ \psi(T)=\dot\psi(T)=0, & x\in\Omega,\\ \psi|_{\Gamma_1}=0,\\ \psi_{\nu_A}|_{\Gamma_0}=z\left[(\phi^{(3)}-\Delta_{\Gamma_g}\dot\phi)h_0-\lambda_T\dot\phi\right], & t\in(0,T),\end{cases} \tag{4.45}$$

where ϕ is produced by (4.25), z and h_0 are given in (4.27) and (4.33), respectively, and

$$\lambda_T=c_T+\frac{1}{2}\sup_{x\in\Gamma_0}|\Delta_{\Gamma_g}h_0|, \tag{4.46}$$

and c_T is given by (4.43).

We define Λ_N: $D(\mathcal{A}_0)\times H^1_{\Gamma_1}(\Omega)\to\left(H^1_{\Gamma_1}(\Omega)\right)'\times L^2(\Omega)$ by

$$\Lambda_N(\phi_0,\phi_1)=(\dot\psi(0),-\psi(0)), \tag{4.47}$$

where $\left(H^1_{\Gamma_1}(\Omega)\right)'$ is the dual space of $H^1_{\Gamma_1}(\Omega)$. Let φ solve the problem (4.25) with the initial data (φ_0,φ_1). After we multiply (4.45) by φ and integrate by parts, we obtain

$$\langle\Lambda_N(\phi_0,\phi_1),(\varphi_0,\varphi_1)\rangle_{L^2(\Omega)\times L^2(\Omega)}=-\int_{\wp_0}\psi_{\nu_A}\varphi d\wp. \tag{4.48}$$

Let $k\geq 1$ be an integer. Let $\Xi^k_{0N}(\Omega)$ consist of the functions u in $H^k(\Omega)$ with the boundary conditions

$$\begin{cases}\mathcal{A}_0^i u|_{\Gamma_1}=(\mathcal{A}_0^i u)_{\nu_A}|_{\Gamma_0}=0, & 0\leq i\leq l-1, \quad \text{if} \quad k=2l;\\ \mathcal{A}_0^i u|_{\Gamma_1}=(\mathcal{A}_0^j u)_{\nu_A}|_{\Gamma_0}=0, & 0\leq i\leq l,\ 0\leq j\leq l-1, \quad \text{if} \quad k=2l+1,\end{cases} \tag{4.49}$$

and with the norms of $H^k(\Omega)$ where $\mathcal{A}_0$ is given by (4.39).

The smooth controls with the Neumann action are provided by the following

Lemma 4.5 *Let $w \in H^{m+1}(\Omega)\cap H^1_{\Gamma_1}(\Omega)$ be exactly controllable such that the assumption* (1.10) *is true and let Γ_1 be such that* (1.14) *holds. Let $k \geq 1$ be an integer and Λ_N be given by* (4.47) *where ψ is the solution of the problem* (4.45). *Then there exists a $T_0 > 0$ such that for any $T > T_0$, there are $c_1 > 0$ and $c_2 > 0$, which depend on T, satisfying*

$$c_1\|(\phi_0,\phi_1)\|_{H^{k+1}(\Omega)\times H^k(\Omega)} \leq \|\Lambda_N(\phi_0,\phi_1)\|_{H^{k-2}(\Omega)\times H^{k-1}(\Omega)} \leq c_2\|(\phi_0,\phi_1)\|_{H^{k+1}(\Omega)\times H^k(\Omega)},$$
$$\forall\ (\phi_0,\phi_1)\in \Xi^{k+1}_{0N}(\Omega)\times\Xi^k_{0N}(\Omega), \tag{4.50}$$

where for $k = 1$, $H^{k-2}(\Omega)\times H^{k-1}(\Omega) = \left(H^1_{\Gamma_1}(\Omega)\right)'\times L^2(\Omega)$. In particular, Λ_N are isomorphisms from $\Xi^2_{0N}(\Omega)\times\Xi^1_{0N}(\Omega)$ onto $\left(H^1_{\Gamma_1}(\Omega)\right)'\times L^2(\Omega)$ and from $\Xi^{k+1}_{0N}(\Omega)\times\Xi^k_{0N}(\Omega)$ onto $\left(H^{k-2}(\Omega)\cap H^1_{\Gamma_1}(\Omega)\right)\times\left(H^{k-1}(\Omega)\cap H^1_{\Gamma_1}(\Omega)\right)$ for $k\geq 2$, respectively.

Proof. By induction.

Let $k = 1$. Let $(\phi_0,\phi_1)\in\aleph^2_{0N}(\Omega)\times\aleph^1_{0N}(\Omega)$ be given. For any $(\varphi_0,\varphi_1)\in\aleph^2_{0N}(\Omega)\times\aleph^1_{0N}(\Omega)$ given, suppose that φ solves the problem (4.25) with the initial data (φ_0,φ_1). Then $\dot\varphi$ solves the problem (4.25) with the initial $(\varphi_1,\mathcal{A}_0\varphi_0)\in H^1_{\Gamma_1}(\Omega)\times L^2(\Omega)$. By the formula (4.48), we obtain

$$\begin{aligned}
&\left(\dot\psi(0),\varphi_1\right) - (\psi(0),\mathcal{A}_0\varphi_0)\\
&= \langle\Lambda_N(\phi_0,\phi_1),(\varphi_1,\mathcal{A}_0\varphi_0)\rangle_{L^2(\Omega)\times L^2(\Omega)} = -\int_{\wp_0}\psi_{\nu_A}\dot\varphi d\wp\\
&= \Psi_*(\dot\phi,\dot\varphi) + \int_{\Gamma_0} h_0\varphi_1\Delta_{\Gamma_g}\phi_0 d\Gamma + \int_{T-\varepsilon}^T\int_{\Gamma_0}\dot z h_0\dot\varphi\ddot\phi d\Gamma dt\\
&\quad -\int_{\wp_0} z\dot\phi\nabla_{\Gamma_g}h_0(\dot\varphi)d\wp + \frac12\sup_{x\in\Gamma_0}|\Delta_{\Gamma_g}h_0|\int_{\wp_0} z\dot\phi\dot\varphi d\wp.
\end{aligned}\tag{4.51}$$

It follows from (4.51) and Lemma 4.4 that

$$\begin{aligned}
&\left(\dot\psi(0),\varphi_1\right) + (\psi(0),\mathcal{A}_0\varphi_0)\\
&\leq \Psi_*^{1/2}(\dot\phi,\dot\phi)\Psi_*^{1/2}(\dot\varphi,\dot\varphi) + c\|\Delta_{\Gamma_g}\phi_0\|_{H^{-1/2}(\Gamma_0)}\|\varphi_1\|_{H^{1/2}(\Gamma_0)}\\
&\quad + c_\varepsilon\int_{T-\varepsilon}^T\|\Delta\phi\|_{H^{-1/2}(\Gamma_0)}\|\dot\varphi\|_{H^{1/2}(\Gamma_0)}dt + c\int_0^T\|\dot\varphi\|_{H^{1/2}(\Gamma_0)}\|\dot\phi\|_{H^{1/2}(\Gamma_0)}dt\\
&\leq \hat c_T E^{1/2}(\phi_1,\mathcal{A}_0\phi_0)E^{1/2}(\varphi_1,\mathcal{A}_0\varphi_0),
\end{aligned}\tag{4.52}$$

for any $T > T_2+\varepsilon$ and all $(\varphi_0,\varphi_1)\in\aleph^2_{0N}(\Omega)\times\aleph^1_{0N}(\Omega)$, which gives

$$\|\dot\psi(0)\|^2_{\left(H^1_{\Gamma_1}(\Omega)\right)'} + \|\psi(0)\|^2 \leq \hat c_T E(\phi_1,\mathcal{A}_0\phi_0),\quad \forall\ (\phi_0,\phi_1)\in\aleph^2_{0N}(\Omega)\times\aleph^1_{0N}(\Omega). \tag{4.53}$$

Furthermore, letting $(\varphi_0,\varphi_1) = (\phi_0,\phi_1)$ in the identity (4.51) yields, via Lemma 4.4,

$$\begin{aligned}
&\left(\|\dot\psi(0)\|^2_{\left(H^1_{\Gamma_1}(\Omega)\right)'} + \|\psi(0)\|^2\right)^{1/2} E^{1/2}(\varphi_1,\mathcal{A}_0\varphi_0)\\
&\geq \left[\rho_0(T-\varepsilon) - c_{\varepsilon 1}\right]E(\phi_1,\mathcal{A}_0\phi_0) - c_\varepsilon E(\phi_1,\mathcal{A}_0\phi_0)\\
&\geq \left[\rho_0(T-\varepsilon) - c_{\varepsilon 1}\right]E(\phi_1,\mathcal{A}_0\phi_0),
\end{aligned}\tag{4.54}$$

for $T > T_2+\varepsilon$, where the constant $c_{\varepsilon 1}$ may be different from that in Lemma 4.4 but is independent of time t and solutions ϕ.

Combining (4.53) and (4.54), we have obtained a $T_0>0$ such that the inequality (4.50) is true for $k=1$.

We assume that the inequality (4.50) is true for some $k\geq 1$. We shall prove it holds true with k replaced by $k+1$.

Case I Let $k=2l$ for some $l\geq 1$. Firstly, we assume that

$$(\phi_0,\phi_1)\in \aleph_{0N}^{2k+2}(\Omega)\times \aleph_{0N}^{2k+1}(\Omega). \tag{4.55}$$

Suppose that φ solves the problem (4.25) with an initial data $(\varphi_0,\varphi_1)\in \aleph_{0N}^{2k+2}(\Omega)\times \aleph_{0N}^{2k+1}(\Omega)$. Then $\varphi^{(2i)}$ and $\varphi^{(2j+1)}$ solve the problem (4.25) with the initial data $(\mathcal{A}_0^i\varphi_0,\mathcal{A}_0^i\varphi_1)$ and $(\mathcal{A}_0^j\varphi_1,\mathcal{A}_0^{j+1}\varphi_0)$, respectively, for $0\leq i\leq k+1$ and $0\leq j\leq k-1$.

Step 1 The following identity is true.

$$\begin{aligned}
&-\Big(A\nabla(\mathcal{A}_0^{l-1}\dot\psi(0)),\nabla(\mathcal{A}_0^l\varphi_1)\Big)-\Big(\mathcal{A}_0^l\psi(0),\mathcal{A}_0^{l+1}\varphi_0\Big)\\
&=\Psi_*(\phi^{(k+1)},\varphi^{(k+1)})+\frac12\sup_{x\in\Gamma_0}|\Delta_{\Gamma_g}h_0|\int_{\wp_0}z\phi^{(k+1)}\varphi^{(k+1)}d\wp\\
&\quad-\int_{\wp_0}z\varphi^{(k+1)}\nabla_{\Gamma_g}h_0(\phi^{(k+1)})d\wp+\sum_{j=1}^{k-1}\int_{T-\varepsilon}^T\int_{\Gamma_0}z^{(j)}\phi^{(k+2-j)}\varphi^{(k+2)}h_0d\Gamma dt\\
&\qquad+\sum_{j=1}^{k}\int_{T-\varepsilon}^T\int_{\Gamma_0}z^{(j)}\varphi^{(k+1)}\Big(\Delta_{\Gamma_g}\phi^{(k+1-j)}h_0+\lambda_T\phi^{(k+1-j)}\Big)d\Gamma dt.
\end{aligned}\tag{4.56}$$

Proof of (4.56) Using $\varphi^{(2k+1)}$ in place of φ in the formula (4.48), we obtain

$$\begin{aligned}
&\Big(\dot\psi(0),\mathcal{A}_0^k\varphi_1\Big)-\Big(\psi(0),\mathcal{A}_0^{k+1}\varphi_0\Big)=-\int_{\wp_0}\psi_{\nu_A}\varphi^{(2k+1)}d\wp\\
&=-\int_{\wp_0}h_0z\phi^{(3)}\varphi^{(2k+1)}d\wp+\int_{\wp_0}zh_0\varphi^{(2k+1)}\Delta_{\Gamma_g}\dot\phi d\wp+\lambda_T\int_{\wp_0}z\dot\phi\varphi^{(2k+1)}d\wp\\
&=\text{Term }1+\text{ Term }2+\lambda_T\text{ Term }3,
\end{aligned}\tag{4.57}$$

where λ_T is given by (4.46).

We compute the terms in the right-hand side of (4.57) by integrating by parts over $\wp_0=(0,T)\times\Gamma_0$, respectively, as

$$\begin{aligned}
\text{Term }1\ &=\int_{\wp_0}z\phi^{(k+2)}\varphi^{(k+2)}h_0d\wp+\sum_{j=0}^{k-2}(-1)^j\int_{\Gamma_0}\phi^{(j+3)}(0)\varphi^{(2k-j)}(0)h_0d\Gamma\\
&\quad+\sum_{j=1}^{k-1}\int_{T-\varepsilon}^T\int_{\Gamma_0}z^{(j)}\phi^{(k+2-j)}\varphi^{(k+2)}h_0d\Gamma dt\\
&=\int_{\wp_0}z\phi^{(k+2)}\varphi^{(k+2)}h_0d\wp+\sum_{j=1}^{l}\int_{\Gamma_0}\mathcal{A}_0^j\phi_1\mathcal{A}_0^{k+1-j}\varphi_0h_0d\Gamma
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=2}^{l}\int_{\Gamma_0}\mathcal{A}_0^j\phi_0\mathcal{A}_0^{k+1-j}\varphi_1 h_0 d\Gamma\\
&+\sum_{j=1}^{k-1}\int_{T-\varepsilon}^{T}\int_{\Gamma_0} z^{(j)}\phi^{(k+2-j)}\varphi^{(k+2)}h_0 d\Gamma dt;
\end{aligned}
\tag{4.58}
$$

$$
\begin{aligned}
\text{Term 2 } &= \int_{\wp_0} z h_0\varphi^{(k+1)}\Delta_{\Gamma_g}\phi^{(k+1)}d\wp + \sum_{j=1}^{k}\int_{T-\varepsilon}^{T}\int_{\Gamma_0} z^{(j)}\varphi^{(k+1)}\Delta_{\Gamma_g}\phi^{(k+1-j)}h_0 d\Gamma dt\\
&\quad+\sum_{j=0}^{k-1}(-1)^{j+1}\int_{\Gamma_0} h_0\varphi^{(2k-j)}(0)\Delta_{\Gamma_g}\phi^{(j+1)}(0)d\Gamma\\
&= -\int_{\wp_0} z\langle\nabla_{\Gamma_g}\phi^{(k+1)},\nabla_{\Gamma_g}\varphi^{(k+1)}\rangle_g h_0 d\wp - \int_{\wp_0} z\varphi^{(k+1)}\nabla_{\Gamma_g}h_0(\phi^{k+1)})d\wp\\
&\quad+\sum_{j=1}^{l}\int_{\Gamma_0} h_0\mathcal{A}_0^{k-j}\varphi_1\Delta_{\Gamma_g}\mathcal{A}_0^j\phi_0 d\Gamma - \sum_{j=0}^{l-1}\int_{\Gamma_0} h_0\mathcal{A}_0^{k-j}\varphi_0\Delta_{\Gamma_g}\mathcal{A}_0^j\phi_1 d\Gamma\\
&\quad+\sum_{j=1}^{k}\int_{T-\varepsilon}^{T}\int_{\Gamma_0} z^{(j)}\varphi^{(k+1)}\Delta_{\Gamma_g}\phi^{(k+1-j)}h_0 d\Gamma dt;
\end{aligned}
\tag{4.59}
$$

$$
\begin{aligned}
\text{Term 3 } &= \int_{\wp_0} z\phi^{(k+1)}\varphi^{(k+1)}d\wp + \sum_{j=1}^{k}\int_{T-\varepsilon}^{T}\int_{\Gamma_0} z^{(j)}\phi^{(k+1-j)}\varphi^{(k+1)}d\Gamma dt\\
&\quad+\sum_{j=1}^{l}\int_{\Gamma_0}\mathcal{A}_0^j\phi_0\mathcal{A}_0^{k-j}\varphi_1 d\Gamma - \sum_{j=0}^{l-1}\int_{\Gamma_0}\mathcal{A}_0^j\phi_1\mathcal{A}_0^{k-j}\varphi_0 d\Gamma.
\end{aligned}
\tag{4.60}
$$

Moreover, via the problem (4.45), we have, on Γ_0 for $j \ge 0$,

$$
\left(\mathcal{A}_0^j\dot{\psi}(0)\right)_{\nu_A} = \psi_{\nu_A}^{(2j+1)}(0) = (\mathcal{A}_0^{j+2}\phi_0 - \Delta_{\Gamma_g}\mathcal{A}_0^{j+1}\phi_0)h_0 - \lambda_T\mathcal{A}_0^{j+1}\phi_0;
\tag{4.61}
$$

$$
\left(\mathcal{A}_0^j\psi(0)\right)_{\nu_A} = (\mathcal{A}_0^{j+1}\phi_1 - \Delta_{\Gamma_g}\mathcal{A}_0^j\phi_1)h_0 - \lambda_T\mathcal{A}_0^j\phi_1.
\tag{4.62}
$$

We substitute (4.58)-(4.62) into (4.57) to yield

$$
\begin{aligned}
&\left(\dot{\psi}(0),\mathcal{A}_0^k\varphi_1\right) - \left(\psi(0),\mathcal{A}_0^{k+1}\varphi_0\right) = \Psi_*(\phi^{(k+1)},\varphi^{(k+1)})\\
&\quad+\frac{1}{2}\sup_{x\Gamma_0}|\Delta_{\Gamma_g}h_0|\int_{\wp_0} z\phi^{(k+1)}\varphi^{(k+1)}d\wp\\
&\quad-\int_{\wp_0} z\varphi^{(k+1)}\nabla_{\Gamma_g}h_0(\phi^{k+1)})d\wp + \sum_{j=1}^{k-1}\int_{T-\varepsilon}^{T}\int_{\Gamma_0} z^{(j)}\phi^{(k+2-j)}\varphi^{(k+2)}h_0 d\Gamma dt\\
&\quad+\sum_{j=1}^{k}\int_{T-\varepsilon}^{T}\int_{\Gamma_0} z^{(j)}\varphi^{(k+1)}\left(\Delta_{\Gamma_g}\phi^{(k+1-j)}h_0 + \phi^{(k+1-j)}\right)d\Gamma dt\\
&\quad+\sum_{j=0}^{l-1}\int_{\Gamma_0}\left(\mathcal{A}_0^j\psi(0)\right)_{\nu_A}\mathcal{A}_0^{k-j}\varphi_0 d\Gamma - \sum_{j=0}^{l-2}\int_{\Gamma_0}\left(\mathcal{A}_0^j\dot{\psi}(0)\right)_{\nu_A}\mathcal{A}_0^{k-1-j}\varphi_1 d\Gamma.
\end{aligned}
\tag{4.63}
$$

On the other hand, for $(\varphi_0, \varphi_1) \in \aleph_{0N}^{2k+2}(\Omega) \times \aleph_{0N}^{2k+1}(\Omega)$, we obtain via the Green formula

$$
\begin{aligned}
\left(\dot{\psi}(0), \mathcal{A}_0^k \varphi_1\right) &= -\left(A\nabla(\mathcal{A}_0^{l-1}\dot{\psi}(0)), \nabla(\mathcal{A}_0^l \varphi_1)\right) \\
&\quad - \sum_{j=0}^{l-2} \int_{\Gamma_0} \left(\mathcal{A}_0^j \dot{\psi}(0)\right)_{\nu_A} \mathcal{A}_0^{k-1-j} \varphi_1 d\Gamma;
\end{aligned} \tag{4.64}
$$

$$
-\left(\psi(0), \mathcal{A}_0^{k+1}\varphi_0\right) = -\left(\mathcal{A}_0^l \psi(0), \mathcal{A}_0^{l+1}\varphi_0\right) + \sum_{j=0}^{l-1} \int_{\Gamma_0} \left(\mathcal{A}_0^j \psi(0)\right)_{\nu_A} \mathcal{A}_0^{k-j}\varphi_0 d\Gamma. \tag{4.65}
$$

After substituting (4.64) and (4.65) into the left-hand side of the identity (4.63) and eliminating the same terms from the both sides, we obtain the identity (4.56).

Step 2 We have

$$
\|(\dot{\psi}(0), \psi(0))\|^2_{H^{k-1}(\Omega)\times H^k(\Omega)} \geq c_1 \|(\phi_0, \phi_1)\|^2_{H^{k+2}(\Omega)\times H^{k+1}(\Omega)}, \tag{4.66}
$$

for all $(\phi_0, \phi_1) \in \aleph_{0N}^{k+2}(\Omega) \times \aleph_{0N}^{k+1}(\Omega)$ and T large.

Proof of (4.66) Replace ϕ with $\phi^{(k)}$ in the inequality (4.44) and obtain

$$
c_{\varepsilon 2} T E(\mathcal{A}_0^l \phi_1, \mathcal{A}_0^{l+1}\phi_0) \geq \Psi_*(\phi^{(k+1)}, \phi^{(k+1)}) \geq [\rho_0(T-\varepsilon) - c_{\varepsilon 1}]\, E(\mathcal{A}_0^l \phi_1, \mathcal{A}_0^{l+1}\phi_0), \tag{4.67}
$$

for all $(\phi_0, \phi_1) \in \aleph_{0N}^{k+2}(\Omega) \times \aleph_{0N}^{k+1}(\Omega)$.

We let $(\varphi_0, \varphi_1) = (\phi_0, \phi_1)$ in the identity (4.56) and observe that

$$
\int_{\wp_0} z\phi^{(k+1)} \nabla_{\Gamma_g}(\phi^{(k+1)}) d\wp = -\frac{1}{2}\int_{\wp_0} [\phi^{(k+1)}]^2 \Delta_{\Gamma_g} h_0 d\wp; \tag{4.68}
$$

$$
\begin{aligned}
&\Big| \sum_{j=1}^{k-1} \int_{T-\varepsilon}^{T} \int_{\Gamma_0} z^{(j)} \phi^{(k+2-j)} \phi^{(k+2)} h_0 d\Gamma dt \Big| \\
&\leq c_\varepsilon \sum_{j=1}^{k-2} \sup_{T-\varepsilon \leq t \leq T} \|\phi^{(k+2-j)}\|_{L^2(\Gamma_0)} \|\phi^{(k+1)}\|_{L^2(\Gamma_0)} \leq c_\varepsilon E(\mathcal{A}_0^l \phi_1, \mathcal{A}_0^{l+1}\phi_0);
\end{aligned} \tag{4.69}
$$

$$
\begin{aligned}
&\Big| \sum_{j=1}^{k} \int_{T-\varepsilon}^{T} \int_{\Gamma_0} z^{(j)} \phi^{(k+1)} \left(\Delta_{\Gamma_g}\phi^{(k+1-j)} h_0 + \lambda_T \phi^{(k+1-j)}\right) d\Gamma dt \Big| \\
&\leq c_\varepsilon \sum_{j=1}^{k} \sup_{T-\varepsilon \leq t \leq T} \|\Delta_{\Gamma_g}\phi^{(k+1-j)}\|_{H^{-1/2}(\Gamma_0)} \|\phi^{(k+1)}\|_{H^{1/2}(\Gamma_0)} \\
&\quad + \lambda_T c_\varepsilon \sum_{j=1}^{k} \sup_{T-\varepsilon \leq t \leq T} \|\phi^{(k+1-j)}\|_{H^{1/2}(\Gamma_0)} \|\phi^{(k+1)}\|_{H^{-1/2}(\Gamma_0)} \\
&\leq c_\varepsilon E(\mathcal{A}_0^l \phi_1, \mathcal{A}_0^{l+1}\phi_0) + \lambda_T c_\varepsilon E(\mathcal{A}_0^l \phi_0, \mathcal{A}_0^l \phi_1).
\end{aligned} \tag{4.70}
$$

We then obtain by setting $(\varphi_0, \varphi_1) = (\phi_0, \phi_1)$ in (4.56) and via (4.63)-(4.70)

$$\begin{aligned}
&\|A\nabla(\mathcal{A}_0^{l-1}\dot{\psi}(0))\|^2 + \|\mathcal{A}_0^l\psi(0)\|^2 + \lambda_T c_\varepsilon E(\mathcal{A}_0^l\phi_0, \mathcal{A}_0^l\phi_1) \\
&\geq [\rho_0(T-\varepsilon) - c_{\varepsilon 1}]\, E(\mathcal{A}_0^l\phi_1, \mathcal{A}_0^{l+1}\phi_0).
\end{aligned} \tag{4.71}$$

Next, the inductive assumption that the inequality (3.28) holds for k implies that for T large there is $c > 0$ such that

$$c\|(\dot{\psi}(0), \psi(0))\|^2_{H^{k-2}(\Omega)\times H^{k-1}(\Omega)} \geq E(\mathcal{A}_0^l\phi_0, \mathcal{A}_0^l\phi_1). \tag{4.72}$$

Combining (4.71) and (4.72) yields that the inequality (4.66) is true for all $(\phi_0, \phi_1) \in \aleph_{0N}^{2k+2}(\Omega)\times\aleph_{0N}^{2k+1}(\Omega)$ and T large. Since $\aleph_{0N}^{2k+2}(\Omega)\times\aleph_{0N}^{2k+1}(\Omega)$ is dense in $\aleph_{0N}^{k+2}(\Omega)\times\aleph_{0N}^{k+1}(\Omega)$, then inequality (4.66) is actually true for all $(\phi_0, \phi_1) \in \aleph_{0N}^{k+2}(\Omega)\times\aleph_{0N}^{k+1}(\Omega)$.

Step 3 There is $c_2 > 0$ such that

$$\|\dot{\psi}(0)\|^2_{k-1} + \|\psi(0)\|^2_k \leq c_2\|(\phi_0, \phi_1)\|^2_{H^{k+2}(\Omega)\times H^{k+1}(\Omega)}, \tag{4.73}$$

for all $(\phi_0, \phi_1) \in \aleph_{0N}^{k+2}(\Omega)\times\aleph_{0N}^{k+1}(\Omega)$.

Proof of (4.73) We let $\varphi_1 = 0$ and $\varphi_0 \in \aleph_{0N}^{2k+2}(\Omega)$ in the identity (4.56) and use the inequality (4.63). We obtain

$$\begin{aligned}
|(\mathcal{A}_0^l\psi(0), \mathcal{A}_0^{l+1}\varphi_0)| &\leq \Psi_*^{1/2}(\phi^{(k+1)}, \phi^{(k+1)})\Psi_*^{1/2}(\varphi^{(k+1)}, \varphi^{(k+1)}) \\
&\quad + c\int_0^T \|\phi^{(k+1)}\|_{H^{1/2}(\Gamma_0)}\|\varphi^{(k+1)}\|_{H^{1/2}(\Gamma_0)}dt \\
&\quad + c\sum_{j=1}^{k-1}\int_{T-\varepsilon}^T \|\phi^{(k+2-j)}\|_{H^{1/2}(\Gamma_0)}\|\varphi^{(k+2)}\|_{H^{-1/2}(\Gamma_0)}dt \\
&\quad + c\sum_{j=1}^{k}\int_{T-\varepsilon}^T \|\varphi^{(k+1)}\|_{H^{1/2}(\Gamma_0)}\|\Delta_{\Gamma_g}\phi^{(k+1-j)}\|_{H^{-1/2}(\Gamma_0)}dt \\
&\leq cE^{1/2}(\mathcal{A}_0^l\phi_1, \mathcal{A}_0^{l+1}\phi_0)E^{1/2}(\mathcal{A}_0^l\varphi_1, \mathcal{A}_0^{l+1}\varphi_0) \\
&= cE(\mathcal{A}_0^l\phi_1, \mathcal{A}_0^{l+1}\phi_0)\|\mathcal{A}_0^{l+1}\varphi_0\|,
\end{aligned} \tag{4.74}$$

since $\varphi_1 = 0$. Because $\mathcal{A}_0^{l+1}$: $\aleph_{0N}^{k+2}(\Omega) \to L^2(\Omega)$ is an isomorphism, it follows from (4.74) that

$$\|\mathcal{A}_0^l\psi(0)\|^2 \leq cE((\mathcal{A}_0^l\phi_1, \mathcal{A}_0^{l+1}\phi_0), \tag{4.75}$$

for all $(\phi_0, \phi_1) \in \aleph_{0N}^{k+2}(\Omega)\times\aleph_{0N}^{k+1}(\Omega)$.

Next, by the ellipticity of the operator $\mathcal{A}_0$ and the equation in (4.25), we have

$$\begin{aligned}
&\|\psi(0)\|^2_k \leq c\|\mathcal{A}_0\psi(0)\|^2_{k-2} + c\|\psi_{\nu_A}(0)\|^2_{k-3/2,\Gamma_0} + c\|\psi(0)\|^2_{k-1} \\
&\leq c\|\mathcal{A}_0^2\psi(0)\|^2_{k-4} + c\|\ddot{\psi}_{\nu_A}(0)\|^2_{k-7/2,\Gamma_0} + c\|\psi_{\nu_A}(0)\|^2_{k-3/2,\Gamma_0} + c\|\psi(0)\|^2_{k-1}.
\end{aligned}$$

Repeating this process gives

$$\|\psi(0)\|_k^2 \leq c\|\mathcal{A}_0^l\psi(0)\|^2 + c\sum_{j=0}^{l-1}\|\psi_{\nu_A}^{(2j)}(0)\|_{k-2j-3/2,\Gamma_0}^2 + c\|\psi(0)\|_{k-1}^2. \tag{4.76}$$

We use the boundary control of (4.45) and the equation in the problem (4.25). We obtain

$$\begin{aligned}\sum_{j=0}^{l-1}\|\psi_{\nu_A}^{(2j)}(0)\|_{k-2j-3/2,\Gamma_0}^2 &\leq c\sum_{j=0}^{l-1}\left(\|\phi^{(2j+3)}(0)\|_{k-2j-3/2,\,\Gamma_0}^2 + \|\Delta_{\Gamma_g}\phi^{(2j+1)}(0)\|_{k-2j-3/2,\Gamma_0}^2\right)\\ &\quad + c\sum_{j=0}^{l-1}\|\phi^{(2j+1)}(0)\|_{k-2j-3/2,\,\Gamma_0}^2\\ &\leq c\sum_{j=0}^{l-1}\left(\|\phi^{(2j+3)}(0)\|_{k-2j-1}^2 + \|\mathcal{A}_0\phi^{(2j+1)}(0)\|_{k-2j-1}^2\right)\\ &\leq c\|\phi^{(2l+1)}(0)\|_1^2 \leq cE(\mathcal{A}_0^l\phi_1, \mathcal{A}_0^{l+1}\phi_0). \end{aligned} \tag{4.77}$$

We combine (4.75)-(4.77) and use the inductive assumption

$$\|\psi(0)\|_{k-1}^2 \leq c\|(\phi_0,\phi_1)\|_{H^{k+1}(\Omega)\times H^k(\Omega)}^2,$$

and we have

$$\|\psi(0)\|_k^2 \leq cE(\mathcal{A}_0^l\phi_1, \mathcal{A}_0^{l+1}\phi_0) \leq c\|(\phi_0,\phi_1)\|_{H^{k+2}(\Omega)\times H^{k+1}(\Omega)}^2, \tag{4.78}$$

for all $(\phi_0,\phi_1) \in \aleph_{0N}^{k+2}(\Omega)\times\aleph_{0N}^{k+1}(\Omega)$.

A similar argument establishes the estimate for $\dot{\psi}(0)$.

Case II Let $k = 2l+1$ for some $l \geq 0$. A similar argument shows that the inequality (4.50) holds with k replaced by $k+1$ if it is true for k.

Finally, the lemma follows by induction. ‖

We consider the regularity of the control function in the problem (4.45). Since $\dot{\phi}$ is a lower order term in the the boundary control of (4.45), $\phi^{(3)} - \Delta_{\Gamma_g}\dot{\phi}$ is the principle part of the control. The following lemma relates to the regularity of this principle part.

Lemma 4.6 *Let ϕ solve the problem* (4.25) *with the initial data $(\phi_0,\phi_1) \in D(\mathcal{A}_0)\times H_{\Gamma_1}^1(\Omega)$. Then*

$$\ddot{\phi} - \Delta_{\Gamma_g}\phi \in L^2(\Gamma_0). \tag{4.79}$$

Furthermore, if $(\phi_0,\phi_1) \in \aleph_{0N}^3(\Omega)\times\aleph_{0N}^2(\Omega)$, then

$$\ddot{\phi} - \Delta_{\Gamma_g}\phi \in C\left([0,T], H^{1/2}(\Gamma_0)\right)\cap H^1\left((0,T)\times\Gamma_0\right). \tag{4.80}$$

Proof. Let the Riemann metric g be given by (4.28). Then

$$\mathcal{A}_0\phi = \Delta_g\phi + F(\phi) \quad x \in \Omega, \tag{4.81}$$

where Δ_g is the Laplacian of the metric g and F is a vector field on Ω give by

$$F = \frac{1}{2G} A(x, \nabla w)\nabla G,$$

G being the determinant of $A^{-1}(x, \nabla w)$.

Using the boundary condition $\phi_{\nu_A} = 0$ on Γ_0 and the relation (4.81), we obtain

$$\ddot{\phi} - \Delta_{\Gamma_g}\phi = \frac{1}{|\nu_A|_g^2} D_g^2\phi(\nu_A, \nu_A) + \langle F, \nabla_{\Gamma_g}\phi\rangle_g, \quad x \in \Gamma_0, \tag{4.82}$$

where $D_g^2\phi(\cdot,\cdot)$ is the Hessian of ϕ in the metric g. Since $\|\nabla_{\Gamma_g}\phi\|^2_{L^2(\Gamma_0)} \le cE(\phi_1, \mathcal{A}_0\phi_0)$, to get the relation (4.79) it will suffice to prove

$$D_g^2\phi(\nu_A, \nu_A) \in L^2(\Gamma_0). \tag{4.83}$$

Let H be a vector field on $\overline{\Omega}$ such that

$$H = 0, \quad x \in \Gamma_1; \quad H = \nu_A, \quad x \in \Gamma_0.$$

We set

$$\varphi = H(\phi), \quad x \in \Omega. \tag{4.84}$$

It is easy to check that φ, given by (4.84), solves the problem with the Dirichlet boundary conditions

$$\begin{cases} \ddot{\varphi} = \mathcal{A}_0\varphi + [H, \mathcal{A}_0]\phi, \quad (t,x) \in (0,T) \times \Omega, \\ \varphi|_\Gamma = 0, \quad t \in (0,T), \\ \varphi(0) = H(\phi_0), \quad \dot{\varphi}(0) = H(\phi_1). \end{cases} \tag{4.85}$$

In addition, $(\phi_0, \phi_1) \in D(\mathcal{A}_0) \times H^1_{\Gamma_1}(\Omega)$ implies $(\varphi(0), \dot{\varphi}(0)) \in H^1_0(\Omega) \times L^2(\Omega)$. We use lemma 3.1 to obtain

$$\varphi_{\nu_A} = D_g^2\phi(\nu_A, \nu_A) + \langle \nabla_{\Gamma_g}\phi,\ (D_g)_{\nu_A}\nu_A\rangle_g \in L^2(\Gamma_0),$$

which gives the relation (4.83).

Next, we assume that $(\phi_0, \phi_1) \in \aleph^3_{0N}(\Omega)\times\aleph^2_{0N}(\Omega)$. Then $(\varphi(0), \dot{\varphi}(0)) \in \left(H^2(\Omega) \times H^1_0(\Omega)\right)\times H^1_0(\Omega)$, where φ is given by (4.84). A similar argument as in the proof of Lemma 3.3 shows that

$$\varphi_{\nu_A} \in C\left([0,T], H^{1/2}(\Gamma)\right) \cap H^1\left((0,T) \times \Gamma\right),$$

which implies that the relation (4.80) is true. $\|$

If $(\phi_0, \phi_1) \in \aleph^3_{0N}(\Omega) \times \aleph^2_{0N}(\Omega)$, the relation (4.80) shows that we can find a control φ in $L^2((0,T) \times \Gamma_0)$ to move one state to another in the space $L^2(\Omega) \times H^1_{\Gamma_0}(\Omega)$ by the control scheme in (4.45).

The Proof of Theorem 4.1 By Lemma 4.5, there is $T_0 > 0$ such that for any $T > T_0$, Λ_N: $\aleph^{m+3}_{0N}(\Omega) \times \aleph^{m+2}_{0N}(\Omega) \to H^m(\Omega) \times H^{m+1}(\Omega)$ is an isomorphism. For any $(v_0, v_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ there is a unique $(\phi_0, \phi_1) \in \aleph^{m+3}_{0N}(\Omega) \times \aleph^{m+2}_{0N}(\Omega)$ such that the solution of the problem (4.22) satisfies (4.23) under the control action

$$\varphi = z\left[(\phi^{(3)} - \Delta_{\Gamma_g}\dot{\phi})h_0 - \lambda_T\dot{\phi}\right], \quad x \in \Gamma_0, \tag{4.86}$$

where ϕ solves the problem (4.25).

To complete the proof, we need to verify $\varphi \in \tilde{\mathcal{X}}^m_{0N}(T)$. Indeed, the relation $(\phi_0, \phi_1) \in \aleph^{m+3}_{0N}(\Omega) \times \aleph^{m+2}_{0N}(\Omega)$ implies that $(\phi^{(m)}(0), \phi^{(m+1)}(0)) \in \aleph^3_{0N}(\Omega) \times \aleph^2_{0N}(\Omega)$. It follows from Lemma 4.6 that $\phi^{(m+2)} - \Delta_{\Gamma_g}\phi^{(m)} \in C\left([0,T], H^{1/2}(\Gamma)\right) \cap H^1((0,T) \times \Gamma)$ which yields $\varphi \in \tilde{\mathcal{X}}^m_{0N}(T)$.

5 Globally exact controllability; Geometrical conditions

The Proof of Theorem 1.3 By Theorem 1.2 and the compactness principle it will suffice to prove that w_α: $[0,1] \to H^m(\Omega)$ is continuous in $\alpha \in [0,1]$.

It is readily seen that $v_\alpha = \frac{\partial}{\partial\alpha}w_a$ is the solution of the following linear, elliptic problem

$$\begin{cases} \sum_{ij=1}^n a_{ij}(x, \nabla w_\alpha)v_{\alpha x_i x_j} + \sum_{l=1}^n \left[\sum_{ij=1}^n a_{ijy_l}(x, \nabla w_\alpha)w_{\alpha x_i x_j} + b_{y_l}(x, \nabla w_\alpha)\right] v_{\alpha x_l} = 0, \\ v_\alpha|_\Gamma = w|_\Gamma, \end{cases} \tag{5.1}$$

for each $\alpha \in [0,1]$, and, in addition, by the maximum principle for the above problem (5.1),

$$\sup_{x\in\Omega} |\frac{\partial}{\partial\alpha}w_\alpha| \le \sup_{x\in\Gamma}|w|. \tag{5.2}$$

Let

$$B(\alpha)v = \sum_{ij=1}^n a_{ij}(x, \nabla w_\alpha)v_{x_i x_j}, \quad v \in H^2(\Omega), \quad \alpha \in [0,1]. \tag{5.3}$$

By the uniform bound (1.17), the ellipticity of the operator $B(\alpha_0)$, and the estimate (5.2), we have

$$\begin{aligned} &\|w_\alpha - w_{\alpha_0}\|_m \\ &\le c\|B(\alpha_0)(w_\alpha - w_{\alpha_0})\|_{m-2} + c|\alpha - \alpha_0|\|w\|_{m-1/2,\,\Gamma} + c\|w_\alpha - w_{\alpha_0}\| \\ &\le c\|B(\alpha_0)(w_\alpha - w_{\alpha_0})\|_{m-2} + c|\alpha - \alpha_0|\left(\|w\|_{m-1/2,\,\Gamma} + \sup_{x\in\Gamma}|w|\right). \end{aligned} \tag{5.4}$$

Next, let us estimate $\|\mathcal{B}(\alpha_0)(w_\alpha - w_{\alpha_0})\|_{m-2}$.

$[B(\alpha_0) - B(\alpha)]w_\alpha$ and $b(x, \nabla w_\alpha) - b(x, \nabla w_{\alpha_0})$ can be written as sums of some terms of the form, respectively,

$$f(x, \nabla w_\alpha, \nabla w_{\alpha_0})(w_{\alpha_0 x_l} - w_{\alpha x_l})w_{\alpha x_i x_j}.$$

Applying the estimate (2.10) to the above products gives, via the bound (1.17) and the estimate (5.2),

$$\begin{aligned}\|B(\alpha_0)(w_\alpha - w_{\alpha_0})\|_{m-2} &\leq \|\,(B(\alpha_0) - B(\alpha))\,w_\alpha\|_{m-2} + \|b(x, \nabla w_a) - b(x, \nabla w_{\alpha_0})\|_{m-2}\\ &\leq c\|w_\alpha - w_{\alpha_0}\|_{m-1}\\ &\leq \varepsilon\|w_\alpha - w_{\alpha_0}\|_{m-2} + c_\varepsilon|\alpha - \alpha_0|\sup_{x\in\Gamma}|w|. \end{aligned} \tag{5.5}$$

We obtain the desired result after substituting (5.5) into (5.4). $\|$

The Proof of Theorem 1.6 The same argument as above completes the proof. $\|$

To end this paper, we prove Proposition 1.1.

The Proof of Proposition 1.1 We only need to prove the case of $\kappa > 0$. By Yao [24], Corollary 1.2, if there are $x_0 \in \overline{\Omega}$ and $\gamma > 0$ such that

$$\Omega \subset B_{g_w}(x_0, \gamma), \quad 4\gamma^2\kappa < \pi^2, \tag{5.6}$$

where

$$B_{g_w}(x_0, \gamma) = \{\, x \,|\, x \in \mathcal{R}^n,\ \rho_{g_w}(x_0, x) < \gamma \,\},$$

then the inequality (1.10) is true. To complete the proof, it will suffice to prove that the condition (1.12) implies (5.6). By (1.12), there is a $0 < \gamma_1 < \lambda\pi/(2\sqrt{\kappa})$ such that

$$\Omega \subset B(x_0, \gamma_1). \tag{5.7}$$

For $x \in B(x_0, \gamma_1)$ be given, $r(t) = tx_0 + (1-t)x$ is a curve in $(\mathcal{R}^n, g_w)$ for $0 \leq t \leq 1$ which connects the points x_0 and x. Then

$$\begin{aligned}\rho_{g_w}(x_0, x) &\leq \int_0^1 |\dot{r}(t)|_{g_w} dt = \int_0^1 \langle A^{-1}(x, \nabla w)\dot{r}(t),\ \dot{r}(t)\rangle^{1/2} dt\\ &\leq \frac{1}{\lambda}|x - y| \leq \frac{\gamma_1}{\lambda}, \end{aligned} \tag{5.8}$$

which implies that (5.6) is true with $\gamma = \gamma_1/\lambda$. $\|$

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